Generalized double-logarithmic large-x resummation

Andreas Vogt (University of Liverpool)

mainly with G. Soar, A. Almasy (UoL), S. Moch (DESY), J. Vermaseren (NIKHEF)

- **Hard lepton-hadron processes in higher-order perturbative QCD**
  Large-x / large-N splitting functions P_{ik} and coefficient functions $C_{a,i}$

- **$\ln^n(1-x)$ behaviour of DIS, SIA and non-singlet DY physical kernels**
  All-order predictions for $C_{a,\mathrm{ns}}$, fourth-order $\ln^{6,5,4}(1-x)$ of P_{ik}

MV, arXiv: 0902.2342, 0909.2124; SMVV, 0912.0369

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  All-order predictions for $C_{a,\mathrm{ns}}$, fourth-order $\ln^{6,5,4}(1-x)$ of P_{ik}, $C_{L,\mathrm{g}}$
- Iteration of (next-to) leading-log unfactorized $1/N$ structure functions
  LL resummation of off-diagonal splitting and coefficient functions
- General D-dimensional structure of large-x DIS and SIA amplitudes
  Verification and extension to higher logarithmic accuracy for DIS/SIA

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Perturbative QCD in collider physics

Search for Higgs Boson, new particles : highest possible energies

$\Rightarrow$ $p\bar{p}/pp$ **colliders: Tevatron (2 TeV), LHC (14 TeV . . . some day)**

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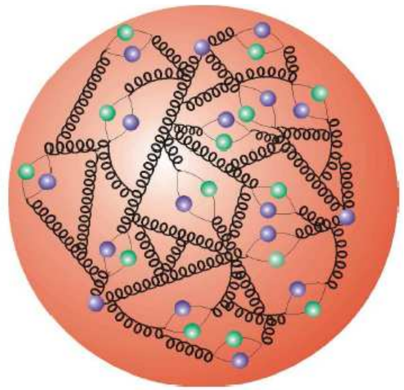
"The good, the bad, and the baryon"

Colliders: wide-band beams of quarks and gluons



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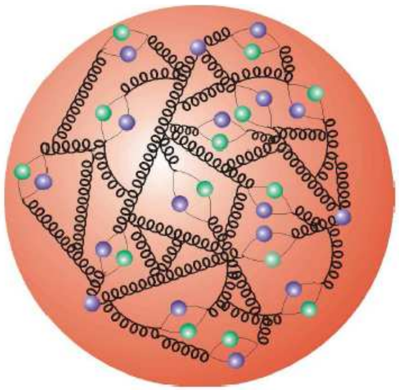
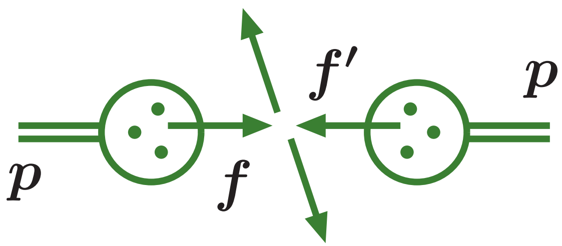


$$\sigma^{pp} = \sum f^p * f'^p * \hat{\sigma}^{ff'}$$



Hard interactions of protons:

parton (q, g) distributions f^p

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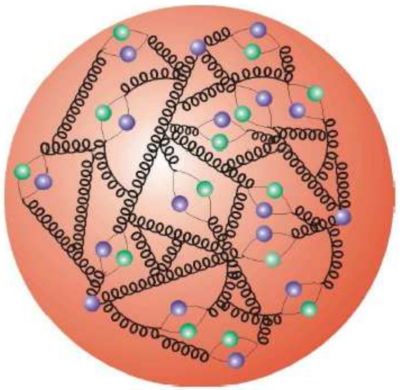

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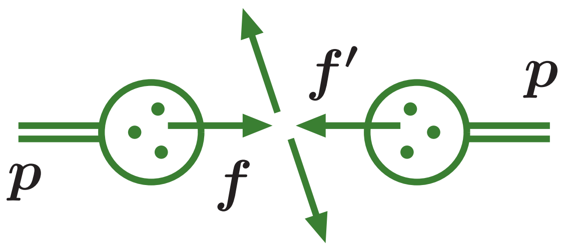

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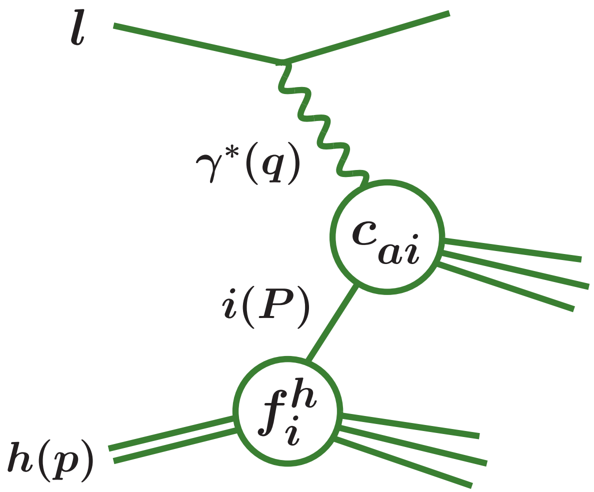
$\Rightarrow$ **Lepton-proton scattering: SLAC** ep, **CERN** μp, νN, **HERA, . . .**

Hard lepton-hadron processes in pQCD (I)

Inclusive deep-inelastic scattering (DIS), semi-incl. l^+l^- annihilation (SIA)



Left $\longrightarrow$ right: DIS, q spacelike, $Q^2 = -q^2$

$$P = \xi p\,,\ f_i^h = \text{parton distributions}$$

Top $\longrightarrow$ bottom: l^+l^-, q timelike, $Q^2 = q^2$

$$p = \xi P\,,\ \text{fragmentation distributions}$$

Drell-Yan (DY) l^+l^- production: bottom $\longrightarrow$ top, 2nd hadron from right ($\{\ldots\}$)

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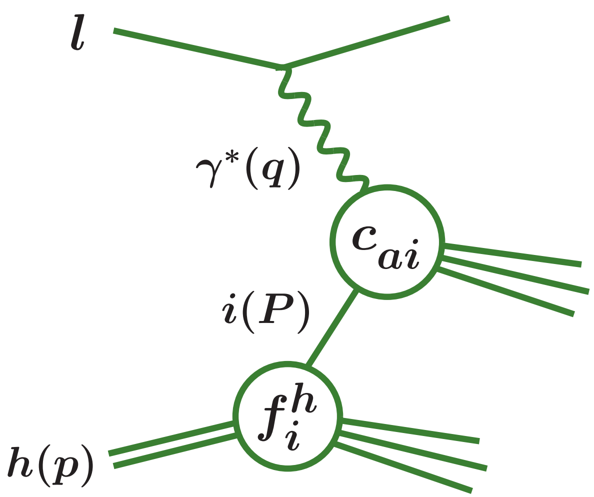

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Structure functions/normalized cross sections F_a: coefficient functions

$$F_a(x, Q^2) \;=\; \Big[C_{a,i\{j\}}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2)\{\otimes f_j^{h'}(\mu^2)\}\Big](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables: $x = Q^2/(2p\cdot q)$ in DIS etc. μ: renorm./mass-fact. scale

Hard lepton-hadron processes in pQCD (II)

Parton/fragmentation distributions f_i : (renorm. group) evolution equations

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) = \left[P_{ik}^{(S,T)}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\right](\xi)$$

$\otimes$ **= Mellin convolution. Initial conditions: fits to reference observables**

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Expansion in α_s: splitting functions P, coefficient fct's c_a of observables

$$\begin{aligned} P \;&=\; \alpha_s\, P^{(0)} + \alpha_s^2\, P^{(1)} + \alpha_s^3\, P^{(2)} + \alpha_s^4\, P^{(3)} + \ldots \\ C_a \;&=\; \underbrace{\alpha_s^{\,n_a}\Big[\, c_a^{(0)} + \alpha_s\, c_a^{(1)}}_{} + \alpha_s^2\, c_a^{(2)} + \alpha_s^3\, c_a^{(3)} + \ldots \Big] \end{aligned}$$

NLO: first real prediction of size of cross sections

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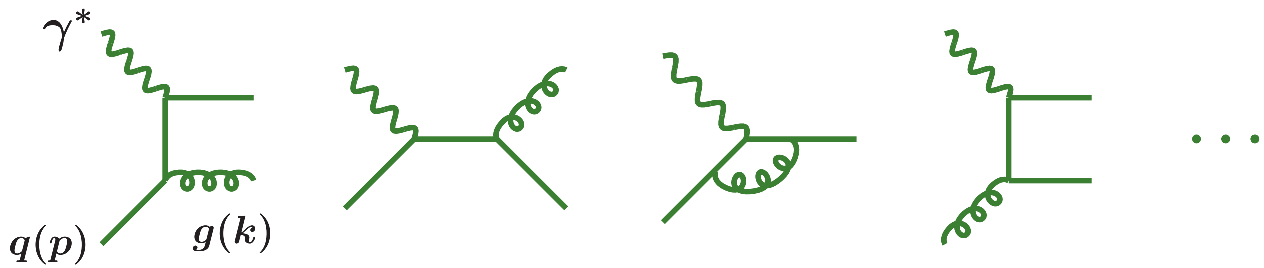
NLO: first real prediction of size of cross sections

NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate of pQCD predictions
N^3LO: for high precision (α_s from DIS), slow convergence (Higgs in $pp/p\bar{p}$)

The 2010 frontier: α_s^4/α_s^3 for DIS/SIA (+ DY) **Baikov, Chetyrkin; MV, . . .**

From mass singularities to splitting functions

First-order DIS:
(inclusive, $\int d^4k$)



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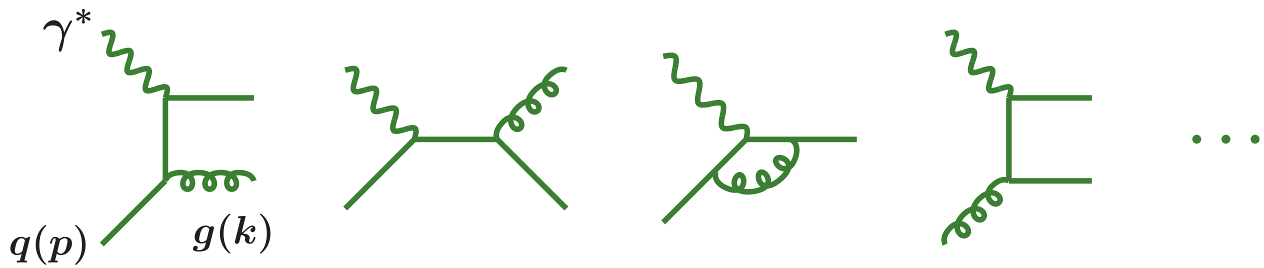

Emissions collinear to the incoming partons ($m_{\mathrm{q,g}} = 0$): denominators

$$(p-k)^2 \;=\; -2|\vec{p}\,||\vec{k}|(1-\cos\vartheta) \;\xrightarrow{\vartheta\to 0}\; -|\vec{p}\,||\vec{k}|\,\vartheta^2 \quad \stackrel{\int d\vartheta}{\Rightarrow} \quad \text{mass singularities}$$

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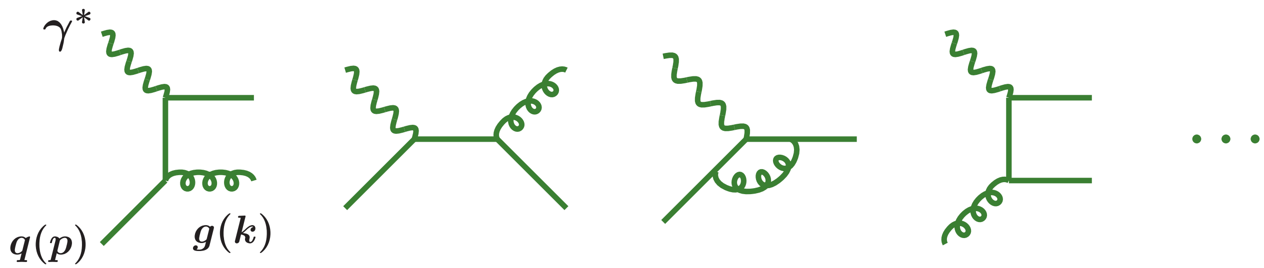

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Regularization (dim. $= 4 - 2\varepsilon$, singularities $\sim 1/\varepsilon$) and mass factorization

$$F_a(Q^2) \;=\; \hat{F}_{a,k}(\alpha_s(Q^2),\varepsilon) \otimes \hat{f}_k \;=\; C_{a,i}(\alpha_s(Q^2)) \otimes \Gamma_{ik}(\alpha_s(Q^2),\varepsilon) \otimes \hat{f}_k$$

$C_{a,i}$ **: coefficient functions of observable a**

Γ_{ik} **: universal $1/\varepsilon$ - poles + . . . (fact. scheme). Usual: $\overline{\text{MS}}$**

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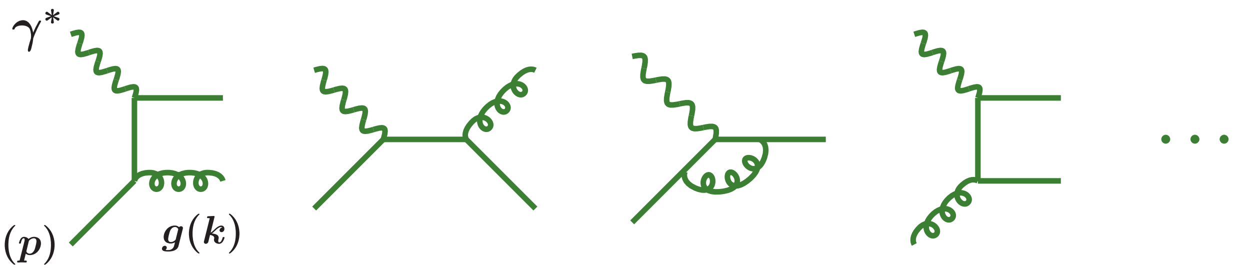

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Renormalized parton distributions f_i : splitting functions P_{ij}

$$\frac{\partial}{\partial \ln Q^2}\,f_i \;=\; \frac{\partial\,\Gamma_{ik}}{\partial \ln Q^2}\otimes \hat{f}_k \;=\; \frac{\partial\,\Gamma_{ik}}{\partial \ln Q^2}\otimes \Gamma^{-1}_{kj}\otimes f_j \;\equiv\; P_{ij}\otimes f_j$$

$\overline{\text{MS}}$ splitting functions at large x / large N

Mellin trf. $f(N) = \int_0^1 dx\, (x^{N-1}\{-1\})\, f(x)_{\{+\}}$: **M-convolutions $\to$ products**

$$\frac{\ln^n(1-x)}{(1-x)_+} \overset{\text{M}}{=} \frac{(-1)^{n+1}}{n+1}\ln^{n+1}N + \ldots\,, \quad \ln^n(1-x) \overset{\text{M}}{=} \frac{(-1)^n}{N}\ln^n N + \ldots$$

Diagonal splitting functions: no higher-order enhancement at N^0, N^{-1}

$$P^{(l-1)}_{\rm qq/gg}(N) = A^{(l)}_{\rm q/g}\ln N + B^{(l)}_{\rm q/g} + C^{(l)}_{\rm q/g}\,\frac{1}{N}\ln N + \ldots\,, \quad A_{\rm g} = C_A/C_F\, A_{\rm q}$$

. . . ; Korchemsky (89); Dokshitzer, Marchesini, Salam (05)

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Off-diagonal: double-log behaviour, colour structure with $C_{AF} = C_A - C_F$

$$C_F^{-1} P_{\mathrm{gq}}^{(l)} \,/\, n_f^{-1} P_{\mathrm{qg}}^{(l)} = \tfrac{1}{N} \ln^{2l} N \;\# C_{AF}^{l} + \tfrac{1}{N} \ln^{2l-1} N\, (\# C_{AF} + \# C_F + \# n_f)\, C_{AF}^{l-1} + \ldots$$

Double logs $\ln^n N$, $l+1 \leq n \leq 2l$ vanish for $C_F = C_A$ ($\rightarrow$ SUSY case)

Aim: obtain, at least, these (next-to) leading terms to all orders l in α_{s}

$\overline{\text{MS}}$ coefficient functions at large x / large N

'Diagonal' [$\mathcal{O}(1)$] coeff. fct's for $F_{2,3,\phi}$ in DIS, $F_{T,A,\phi}$ in SIA, $F_{\rm DY} = \frac{1}{\sigma_0}\frac{d\sigma_{q\bar{q}}}{dQ^2}$

$$C^{(l)}_{2,q/\phi,g/\ldots} \;=\; \#\; \ln^{2l} N + \ldots + N^{-1}(\#\; \ln^{2l-1} N + \ldots) \;+\; \ldots$$

N^0 parts: threshold exponentiation Sterman (87); Catani, Trentadue (89); ...

Exponents known to next-to-next-to-next-to-leading log (N^{3}LL) accuracy - mod. $A^{(4)}$
$\Rightarrow$ highest seven (DIS), six (SIA, DY, Higgs prod.) coefficients known to all orders

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05)
(+ more papers, esp. using SCET, from 2006), SIA: Blümlein, Ravindran (06); MV (09)

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'Off-diagonal' [$\mathcal{O}(\alpha_s)$] quantities: leading N^{-1} double logarithms

$$C^{(l)}_{\phi,\mathrm{q}/2,\mathrm{g}/\ldots} \;=\; N^{-1}(\#\ \ln^{2l-1} N + \#\ \ln^{2l-2} N + \ \ldots)\ +\ \ldots$$

Longitudinal DIS/SIA structure functions [recall: l = order in α_s – 1]

$$C^{(l)}_{\mathrm{L,q}} \;=\; N^{-1}(\#\ \ln^{2l} N + \ \ldots)\ +\ \ldots\ , \quad C^{(l)}_{\mathrm{L,g}} \;=\; N^{-2}(\#\ \ln^{2l} N + \ \ldots)\ +\ \ldots$$

Aim: predict highest N^{-1} [N^{-2} for $C_{\mathrm{L,g}}$] double logarithms to all orders

Non-singlet and singlet physical kernels

Eliminate parton densities from scaling violations of observables ($\mu = Q$)

$$\frac{dF}{d\ln Q^2} \;=\; KF \;\equiv\; \sum_{l=0} {a_{\rm s}}^{l+1} K_l F \;=\; \frac{d\,C}{d\ln Q^2}\,q \;+\; CP\,q$$

$$=\; \Big(\,\beta(a_{\rm s})\,\frac{dC}{da_{\rm s}}C^{\,-1} \;+\; [C,P]\,C^{\,-1} + P\Big)F$$

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Singlet: a) $F = (F_2, F_\phi)$ with large-$m_{\rm top}$ Higgs-exchange DIS

Furmanski, Petronzio (81); ...

Coefficient functions for F_ϕ to order $\alpha_{\rm s}^2 / \alpha_{\rm s}^3$

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni; SMVV (09)

b) $F = (F_2, \widehat{F}_L)$ with $\widehat{F}_L = F_L / a_{\rm s} c_{L,q}^{(0)}$ Catani (96); Blümlein et al. (00)

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NNLO/N^{3}LO: all physical kernels K above single-log enhanced at large N

Conjecture: double-log contributions also vanish at all higher orders in α_s

Non-singlet evolution kernels and predictions

DIS/SIA $a \neq L$ leading-logarithmic kernels, with $p_{\rm qq}(x) = 2/(1-x)_+ - 1 - x$

$$
\begin{aligned}
K_{a,0}(x) &= 2\, C_F\, p_{\rm qq}(x) \\
K_{a,1}(x) &= \ln(1-x)\, p_{\rm qq}(x)\, \left[-2\, C_F \beta_0 \mp 8\, C_F^2 \,\ln x \right] \\
K_{a,2}(x) &= \ln^2(1-x)\, p_{\rm qq}(x)\, \left[\; 2\, C_F \beta_0^2 \pm 12\, C_F^2\, \beta_0 \,\ln x + \mathcal{O}(\ln^2 x) \right] \\
K_{a,3}(x) &= \ln^3(1-x)\, p_{\rm qq}(x)\, \left[-2\, C_F \beta_0^3 \mp 44/3\; C_F^2\, \beta_0^2 \,\ln x + \mathcal{O}(\ln^2 x) \right] \\
K_{a,4}(x) &= \ln^4(1-x)\, p_{\rm qq}(x)\, \left[\; 2\, C_F \beta_0^4 \pm \xi_{K_4} C_F^2\, \beta_0^3 \,\ln x + \mathcal{O}(\ln^2 x) \right]
\end{aligned}
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First term: leading large n_f, all orders via C_2 of Mankiewicz, Maul, Stein (97)

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Conjecture $\Rightarrow$ coefficients of highest three logs from fourth order in α_s,
$\ln^{7,6,5}(1-x)$ at order α_s^4 for $F_{1,2,3}$ in DIS and $F_{\rm T,I,A}$ in SIA etc

Leading terms: $K_1 = K_2$, $K_{\rm T} = K_{\rm I}$ [total ('integrated') fragmentation fct.]

$\Rightarrow$ also three logarithms for space- and timelike F_L: $\ln^{6,5,4}(1-x)$ at α_s^4 etc

Alternative derivation: physical kernels for F_L, agreement non-trivial check

All-order resummation of the $1/N$ terms (I)

For $F_{1,2,3}$, $F_{\mathrm{T,I,A}}$ and F_{DY}, up to terms of order $1/N^2$, with $L \equiv \ln N$

$$C_a(N) - C_a\Big|_{N^0 L^k} \; = \; \frac{1}{N}\left(\left[d_{a,1}^{(1)} L + d_{a,0}^{(1)}\right] a_{\mathrm{s}} + \left[\widetilde{d}_{a,1}^{(2)} L + d_{a,0}^{(2)}\right] a_{\mathrm{s}}^2 + \ldots\right)$$

$$\exp\left\{L h_1(a_{\mathrm{s}} L) + h_2(a_{\mathrm{s}} L) + a_{\mathrm{s}} h_3(a_{\mathrm{s}} L) + \ldots\right\}$$

Exponentiation functions defined by expansions $h_k(a_{\mathrm{s}} L) \equiv \sum_{n=1} h_{kn} (a_{\mathrm{s}} L)^n$

All-order resummation of the $1/N$ terms (I)

For $F_{1,2,3}$, $F_{\mathrm{T,I,A}}$ and F_{DY}, up to terms of order $1/N^2$, with $L \equiv \ln N$

$$C_a(N) - C_a\Big|_{N^0 L^k} = \frac{1}{N}\left(\left[d_{a,1}^{(1)} L + d_{a,0}^{(1)}\right] a_{\mathrm{s}} + \left[\widetilde{d}_{a,1}^{\,(2)} L + d_{a,0}^{(2)}\right] a_{\mathrm{s}}^2 + \ldots\right)$$
$$\exp\left\{L h_1(a_{\mathrm{s}} L) + h_2(a_{\mathrm{s}} L) + a_{\mathrm{s}} h_3(a_{\mathrm{s}} L) + \ldots\right\}$$

Exponentiation functions defined by expansions $h_k(a_{\mathrm{s}} L) \equiv \sum_{n=1} h_{kn} (a_{\mathrm{s}} L)^n$

Coefficients for DIS/SIA (upper/lower sign) relative to $N^0 L^k$ resummation

$$h_{1k} = g_{1k}$$

$g_{lk} =$ coefficients in soft-gluon exponentiation

$$h_{21} = g_{21} + \frac{1}{2}\beta_0 \pm 6\, C_F$$
$$h_{22} = g_{22} + \frac{5}{24}\beta_0^2 \pm \frac{17}{9}\beta_0 C_F - 18\, C_F^2$$
$$h_{23} = g_{23} + \frac{1}{8}\beta_0^3 \pm \left(\frac{\xi_{\mathrm{K}_4}}{8} - \frac{53}{18}\right)\beta_0^2 C_F - \frac{34}{3}\beta_0 C_F^2 \pm 72\, C_F^3$$

First term of h_3 also known, but non-universal within DIS and SIA ($\Leftrightarrow F_L$)

All-order resummation of the $1/N$ terms (II)

For space-like (-) and time-like (+) structure / fragmentation functions F_L

$$C_L^{(\pm)}(N) \;=\; N^{-1}(d_1^{(\pm)} a_s + d_2^{(\pm)} a_s^2 + \ldots)\; \exp\{L h_1(a_s L) + h_2(a_s L) + \ldots\}$$

with

$$h_{11} \;=\; 2\,C_F \;\;,\quad h_{12} \;=\; \frac{2}{3}\,\beta_0\,C_F \;\;,\quad h_{13} \;=\; \frac{1}{3}\,\beta_0^2\,C_F$$

$$h_{21} \;=\; \beta_0 \;+\; 4\,\gamma_e C_F \;-\; C_F \;+\; (4 - 4\,\zeta_2)(C_A - 2C_F)$$

$$h_{22} \;=\; \underbrace{\frac{1}{2}\,(\,\beta_0\,h_{21} + A_2)}_{\text{as } g_{22} \text{ in soft-gluon exp.}} \;-\; \underbrace{8\,(C_A - 2C_F)^2(1 - 3\,\zeta_2 + \zeta_3 + \zeta_2^2\,)}_{\text{Who ordered THIS?}}$$

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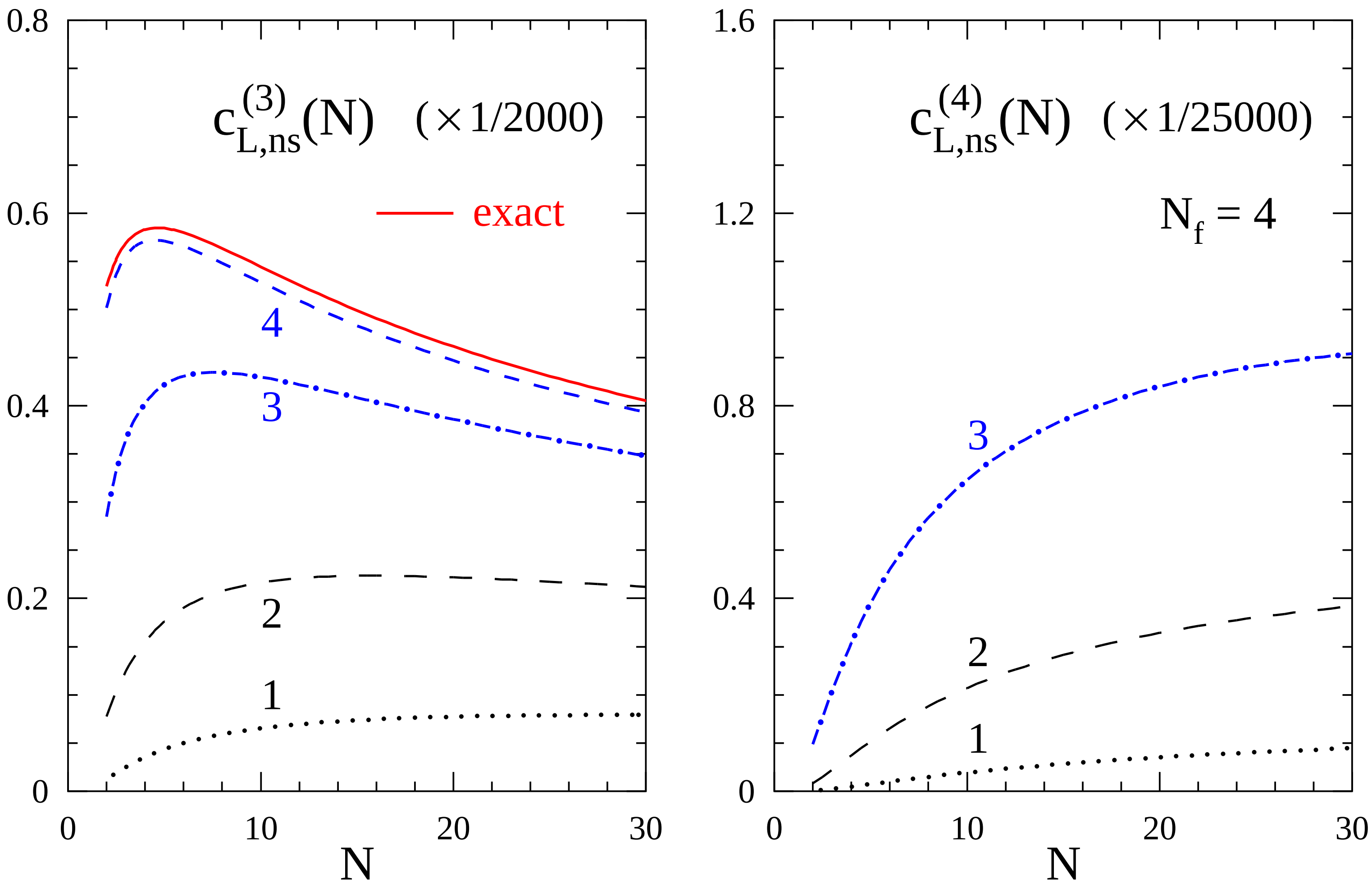
Remarks/questions

- **Less predictive than $N^0 L^k$ exponentiation: nothing new, but A_2, in g_{22}**
- **NLL exponentiation – complete $h_2(a_s L)$ – could be feasible for $F_{a\neq L}$**
- **NNLL exponentiation for $F_{1,2,3}$ etc, NLL for F_L: possible at all?**

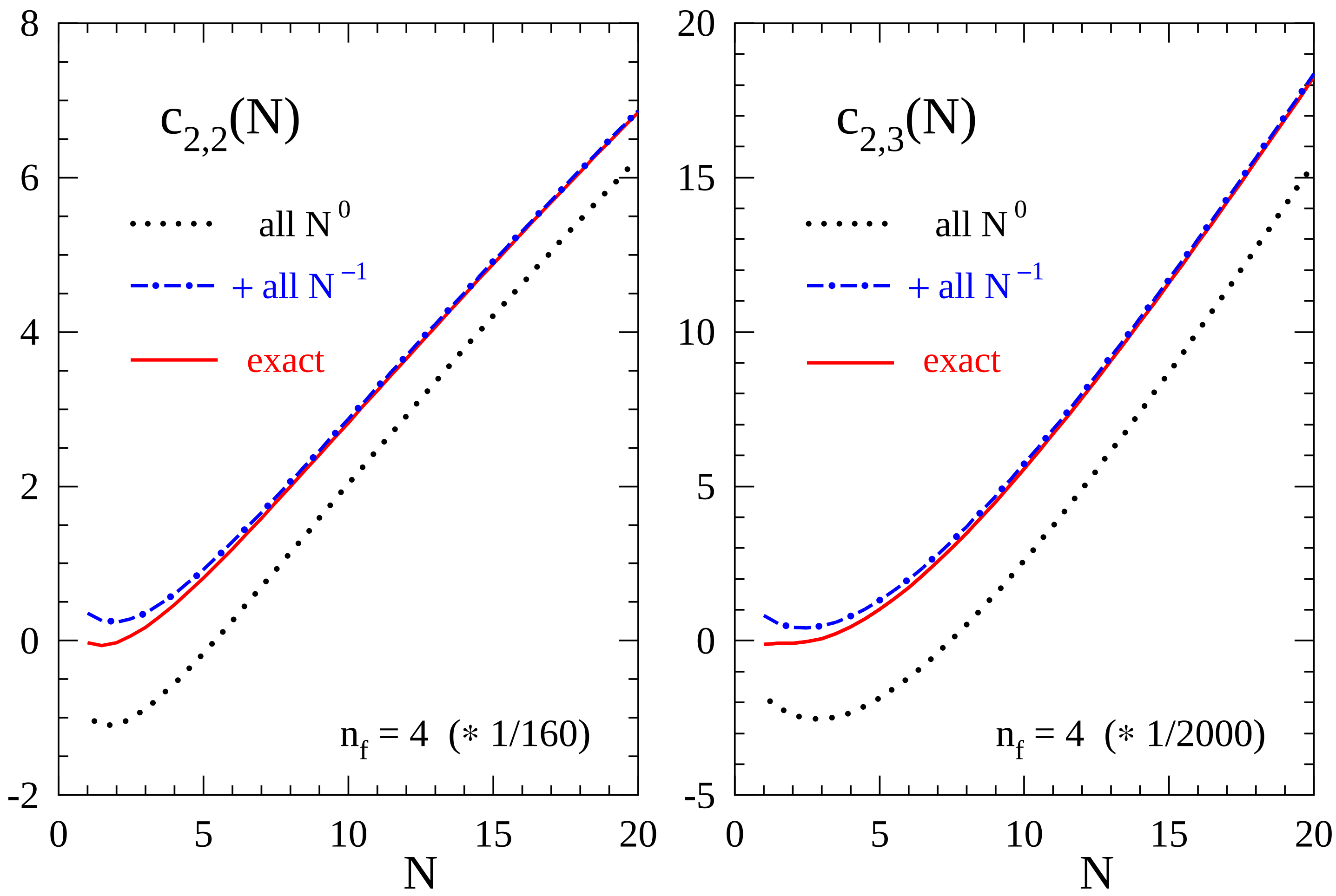
Third- and fourth-order C_L in DIS in N-space



1 = leading log etc. Good α_s^3 approximation by all four N^{-1} logarithms

As usual, cf. small-x: leading logs do not lead. Padé: ≈ 2.0 at $N = 20$

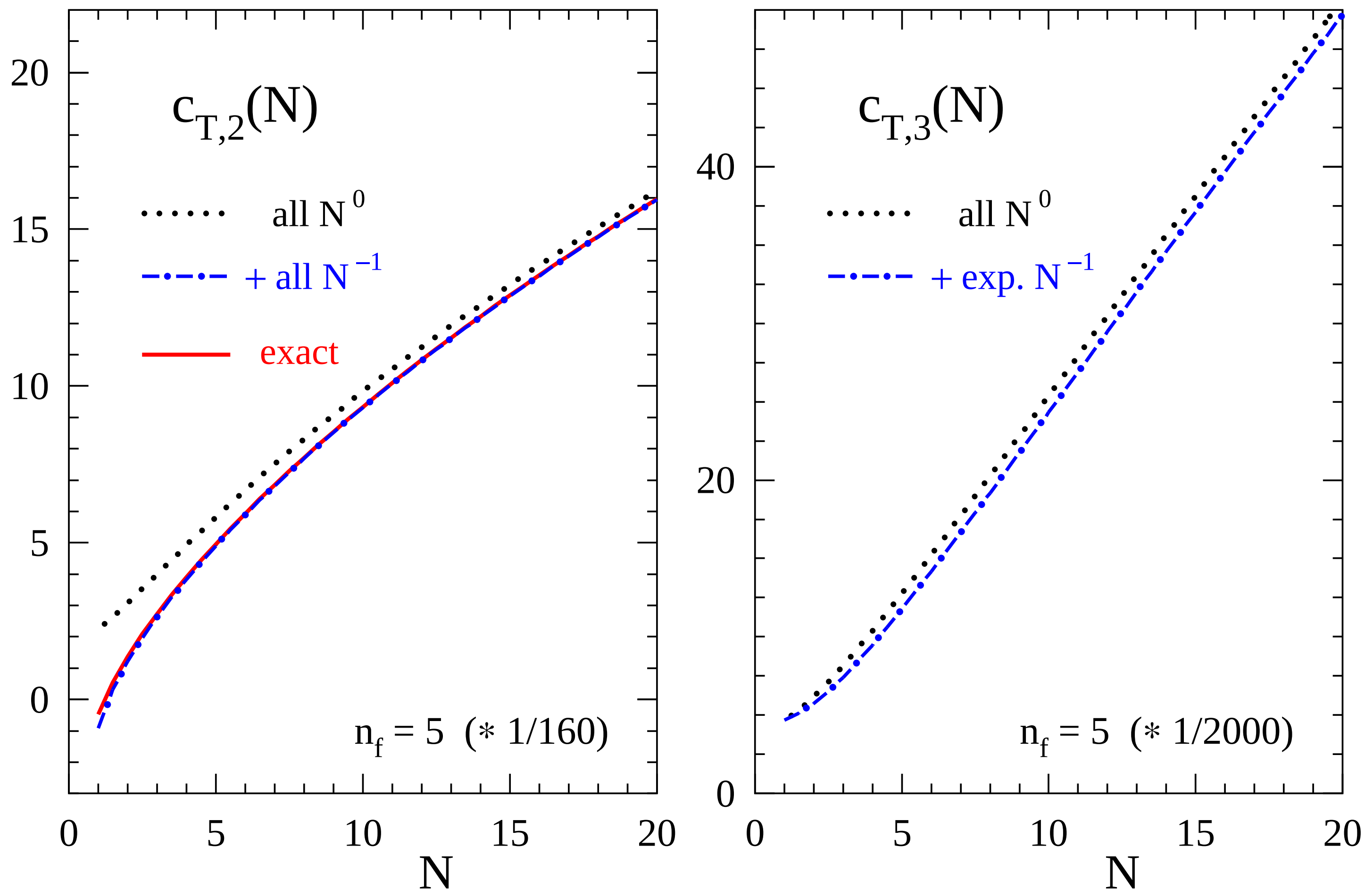
Second- and third-order C_2 in DIS in N-space



N^{-1} terms relevant over full range shown, $\mathcal{O}(N^{-2})$ sizeable only at $N < 5$

Sum of $N^{-1}\ln^k N$ looks almost constant: half of maximum only at $N \simeq 150$

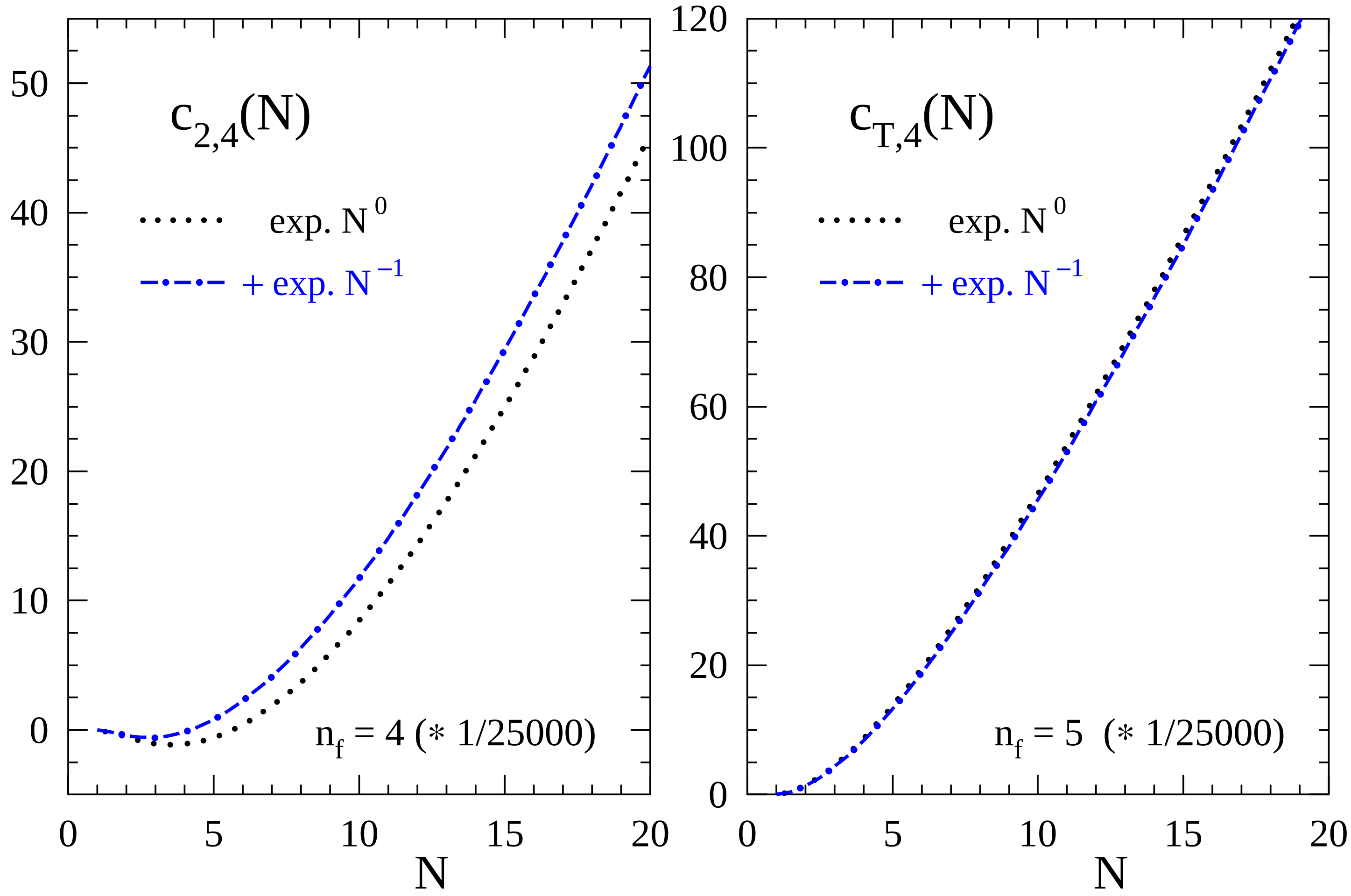
Second- and third-order C_T in SIA in N-space



Larger soft-gluon enhancement, mainly from low powers of $\ln N$ (neg. in DIS)

Compensation between $N^{-1} \ln^k N$ terms (unknown for $k = 0, 1$ at order α_s^3)

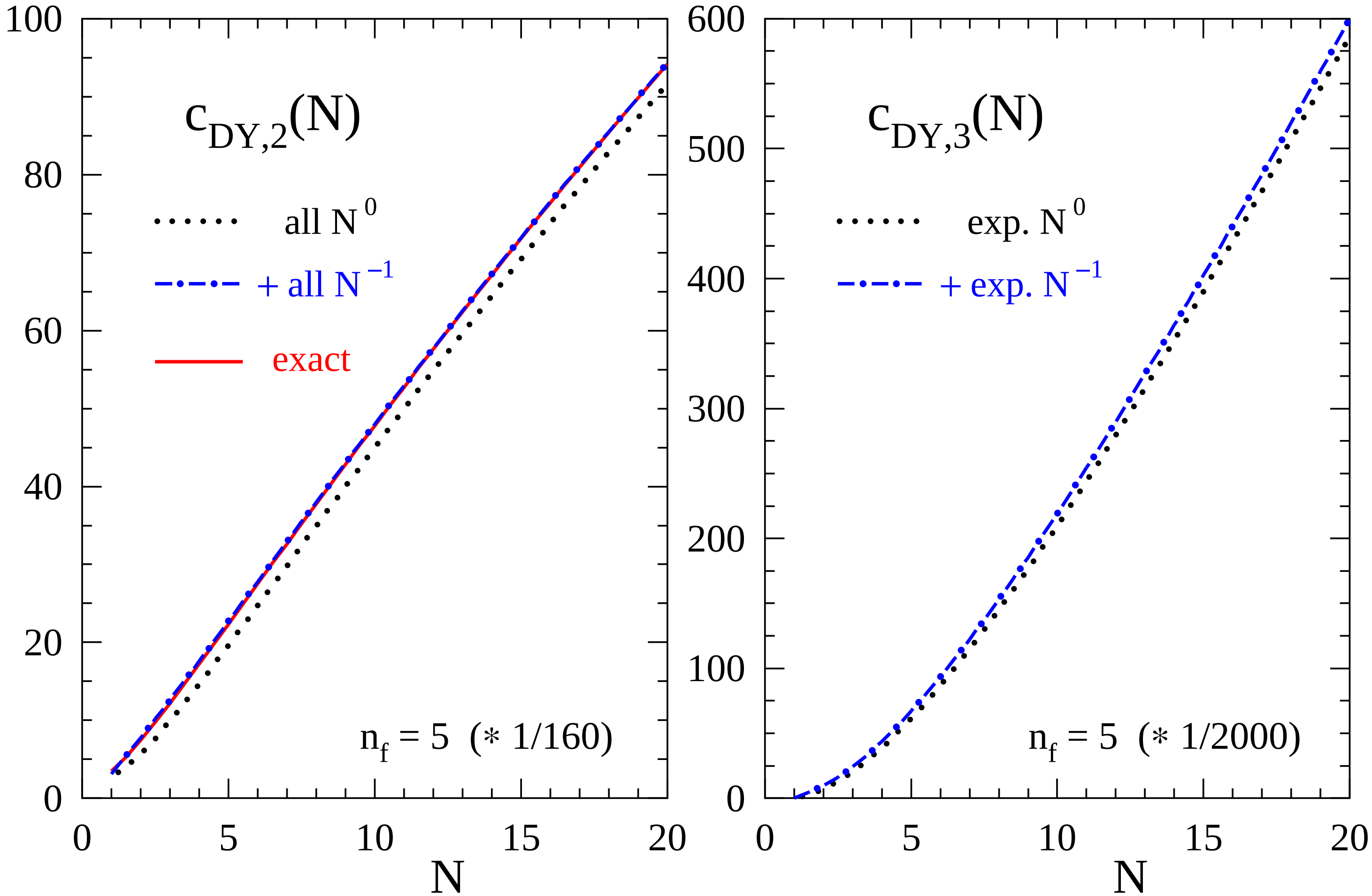
Fourth-order C_2 (DIS) and C_T (SIA) at large N



Exp. N^0: 7 of 8 logs, exp. N^{-1}: 4 of 7 logs – ξ_{K_4} numerically suppressed

N^{-1} contributions again relevant for F_2, but small for F_T at least at $N > 5$

Second- and third-order C_{DY} in N-space



Exp. N^0: all logs, exp. N^{-1}: 3 of 5 logs – ξ_{DY_3} numerically insignificant

N^{-1} contributions small down to even lower moments than in the SIA case

Singlet results: α_s^4 splitting function $P_{qg}^{(3)}(x)$

3-loop coefficient functions + single-log $K_{2\phi,\phi 2}^{(3)}$ $\Rightarrow$ predictions for $P_{qg,gq}^{(3)}$

$$C_{AF} \equiv C_A - C_F$$

$$\begin{aligned}
P_{qg}^{(3)}(x) \;=\;& \ln^6(1-x)\cdot 0 \\
&+ \ln^5(1-x)\left[\frac{22}{27}\,C_{AF}^3 n_f - \frac{14}{27}\,C_{AF}^2 C_F n_f + \frac{4}{27}\,C_{AF}^2 n_f^2\right] \\
&+ \ln^4(1-x)\left[\left(\frac{293}{27} - \frac{80}{9}\,\zeta_2\right) C_{AF}^3 n_f + \left(\frac{4477}{16} - 8\zeta_2\right) C_{AF}^2 C_F n_f \right. \\
&\quad \left. - \frac{13}{81}\,C_{AF} C_F^2 n_f - \frac{116}{81}\,C_{AF}^2 n_f^2 + \frac{17}{81}\,C_{AF} C_F n_f^2 - \frac{4}{81}\,C_{AF} n_f^3\right] \\
&+ \mathcal{O}\left(\ln^3(1-x)\right)
\end{aligned}$$

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$$+ \mathcal{O}\left(\ln^3(1-x)\right)$$

- Vanishing of the coefficient of the leading term at order α_s^4 :
  accidental (??) cancellation of contributions, for all four splitting fct's
- Remaining terms vanish in the supersymmetric case $C_A = C_F (= n_f)$
  Nontrivial check: same as for $P_{qg}^{(2)}$, not obvious from above construction

This prediction + single-logarithmic $K_{2L}^{(3)}$ $\Rightarrow$ $(1-x)\ln^{6,4,3}(1-x)$ of $c_{L,g}^{(3)}$

Threshold logarithms before factorization (I)

Unfactorized partonic structure functions in $D = 4 - 2\varepsilon$ dimensions

$$T_{a,j} = \widetilde{C}_{a,i}\, Z_{ij}\;, \quad -\gamma \equiv P = \frac{dZ}{d\ln Q^2}\, Z^{-1}\;, \quad \frac{da_s}{d\ln Q^2} = -\varepsilon a_s + \beta_{D=4}$$

a_s^n**:** $\varepsilon^{-n} \ldots \varepsilon^{-2}$**: lower-order terms,** ε^{-1}**:** n**-loop splitting functions + . . . ,**
ε^{0}**:** n**-loop coefficient fct's + . . . ,** ε^{k}, $0 < k < l$**: required for order** $n+l$

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a_s^n: $\varepsilon^{-n} \ldots \varepsilon^{-2}$: lower-order terms, ε^{-1}: n-loop splitting functions + ...,
ε^{0}: n-loop coefficient fct's + ..., ε^{k}, $0 < k < l$: required for order $n+l$

N^0 and N^{-1} transition functions Z to next-to-leading log (NLL) accuracy

$$Z\Big|_{a_s^n} = \frac{1}{\varepsilon^n}\frac{\gamma_0^{n-1}}{n!}\Big[\gamma_0 - \frac{\beta_0}{2}n(n-1)\Big] + \sum_{l=1}^{n-1}\frac{1}{\varepsilon^{n-l}}\sum_{k=1}^{n-l-1}\gamma_0^{n-l-k-1}\gamma_l\gamma_0^k\,\frac{(l+k)!}{n!\,l!}$$
$$-\frac{\beta_0}{2}\sum_{l=1}^{n-2}\frac{1}{\varepsilon^{n-l}}\sum_{k=1}^{n-l-2}\gamma_0^{n-l-k-2}\gamma_l\gamma_0^k\,\frac{(l+k)!}{n!\,l!}\big(n(n-1)-l(l+k+1)\big)$$

$+$ **NNLL contributions (explicit expressions)** $+ \ \ldots$

ε^{-n+l} off-diagonal entries: contributions up to $N^{-1}\ln^{n+l-1} N$

Diagonal cases: γ_0 only for N^0 part, second term with $l=1$ for N^{-1} NLL

Threshold logarithms before factorization (II)

D-dimensional coefficient functions $\widetilde{C}_a$: finite for $\varepsilon \longrightarrow 0$

$$\widetilde{C}_{a,i} = 1_{\text{(diagonal cases)}} + \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} a_{\mathsf{s}}^{n} \, \varepsilon^{l} \, c_{a,i}^{(n,l)}$$

$c_{a,i}^{(n,l)}$: l additional factors $\ln N$ relative to $c_{a,i}^{(n,0)} \equiv c_{a,i}^{(n)}$ discussed above

Full N^{m}LO calc. of $T_{a,j}$: highest $m+1$ powers of ε^{-1} to all orders in α_{s}

Extension to all powers of ε: all-order resummation of highest $m+1$ logs

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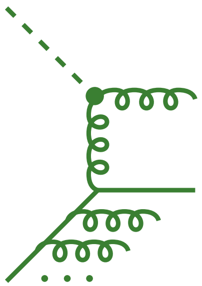
Example: Leading-log (LL) $1/N$ terms of $T_{\phi,\mathrm{q}}^{(n)}$ and $T_{2,\mathrm{g}}^{(n)}$, with $L \equiv \ln N$

$$\frac{1}{C_F} T_{\phi,\mathrm{q}}^{(n)} = \frac{1}{n_f} T_{2,\mathrm{g}}^{(n)} = \frac{L^{n-1}}{N \varepsilon^{n}} \sum_{k=0}^{\infty} (\varepsilon L)^{k} \mathcal{L}_{n,k} \left(C_F^{\,n-1} + C_F^{\,n-2} C_A + \ldots + C_A^{\,n-1} \right)$$

to all orders in ε (calc. + D-dim. structure), with same coefficients $\mathcal{L}_{n,k}$

$\Rightarrow$ all-order relation for one colour structure of either amplitude sufficient

All-order off-diagonal leading-log amplitudes



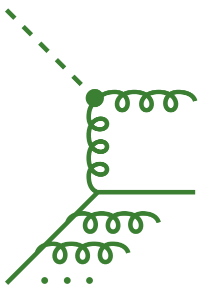
$$T^{(n)}_{\phi,\mathrm{q}}\Big|_{C_F\ \mathrm{only}} \stackrel{\mathrm{LL}}{=} \frac{1}{n}\, T^{(1)}_{\phi,\mathrm{q}}\, \underbrace{T^{(n-1)}_{2,\mathrm{q}}}_{\frac{1}{(n-1)!}\,(T^{(1)}_{2,\mathrm{q}})^{n-1}} \stackrel{\mathrm{LL}}{=} \frac{1}{n!}\, T^{(1)}_{\phi,\mathrm{q}}\, (T^{(1)}_{2,\mathrm{q}})^{n-1}$$

Three-loop diagram calculation + $P^{(3)}_{\mathrm{gq}} \stackrel{\mathrm{LL}}{=} 0$ + general mass factorization:

first four powers in ε known at any order. Rest $\longrightarrow$ higher-order predictions

$$T_{\phi,\mathrm{q}}\Big|_{C_F\ \mathrm{only}} \stackrel{\mathrm{LL}}{=} T^{(1)}_{\phi,\mathrm{q}}\, \frac{\exp(a_{\mathrm{s}} T^{(1)}_{2,\mathrm{q}}) - 1}{T^{(1)}_{2,\mathrm{q}}}$$

All-order off-diagonal leading-log amplitudes



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Exact D-dimensional leading-log expressions for the one-loop amplitudes

$$T^{(1)}_{\phi,\mathrm{q}} \overset{\mathrm{LL}}{=} -2C_F\, \frac{1}{\varepsilon}(1-x)^{-\varepsilon} \overset{\mathrm{M}}{=} -\frac{2C_F}{N}\, \frac{1}{\varepsilon} \exp(\varepsilon \ln N)$$

$$T^{(1)}_{2,\mathrm{q}} \overset{\mathrm{LL}}{=} -4C_F\, \frac{1}{\varepsilon}(1-x)^{-1-\varepsilon} + \text{virtual} \overset{\mathrm{M}}{=} 4C_F\, \frac{1}{\varepsilon^2}(\exp(\varepsilon \ln N) - 1)$$

$\Rightarrow$ leading-log expression for $T_{\phi,\mathrm{q}}$ and $T_{2,\mathrm{g}}$ completely determined

Leading-log splitting and coefficient functions

Expansions and iterative mass factorization to 'any' order [done in FORM]

$\Rightarrow$ All-order expressions for LL off-diagonal splitting and coefficient fct's

$$\boxed{P_{\mathrm{qg}}^{\mathrm{LL}}(N,\alpha_{\mathrm{s}}) \;=\; \frac{n_f}{N}\,\frac{\alpha_{\mathrm{s}}}{2\pi}\sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,\tilde{a}_{\mathrm{s}}^{\,n}\,, \quad \tilde{a}_{\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{\pi}\,(C_A - C_F)\ln^2 N}$$

Bernoulli numbers B_n: zero for odd $n \geq 3$ $\Rightarrow$ $P_{\mathrm{gq}}^{(3)}(N) \stackrel{\mathrm{LL}}{=} 0$ not accidental

$$B_0 = 1,\; B_1 = -\frac{1}{2},\; B_2 = \frac{1}{6},\; B_4 = -\frac{1}{30},\; B_6 = \frac{1}{42},\; \ldots,\; B_{12} = -\frac{691}{2730},\; \ldots$$

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$$\boxed{C^{\mathrm{LL}}_{2,\mathrm{g}} \;=\; \frac{1}{2N\ln N}\,\frac{n_f}{C_A - C_F}\,\{\exp(2C_F a_{\mathrm{s}}\ln^2 N)\,\mathcal{B}_0(\tilde{a}_{\mathrm{s}}) - \exp(2C_A a_{\mathrm{s}}\ln^2 N)\}}$$

$\exp(\ldots)$: **LL soft-gluon exponentials** **Parisi; Curci, Greco; Amati et al. (80)**

$$\mathcal{B}_0(x) \;=\; \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,x^n$$

$P^{\mathrm{LL}}_{\mathrm{gq}}, C^{\mathrm{LL}}_{\phi,\mathrm{q}}$: same functions but with $C_F \leftrightarrow C_A$ (also in $\tilde{a}_{\mathrm{s}}$), then $n_f \to C_F$

First properties of the new $\mathcal{B}$-functions

Relation between even-n Bernoulli numbers and the Riemann ζ-function

$$\mathcal{B}_0(x) = 1 - \frac{x}{2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\,\zeta_{2n}\left(\frac{x}{2\pi}\right)^{2n}$$

$\mathcal{B}_0(2\pi i)$ numerically known (Wolfram MathWorld, Sloane's A093721), no closed form

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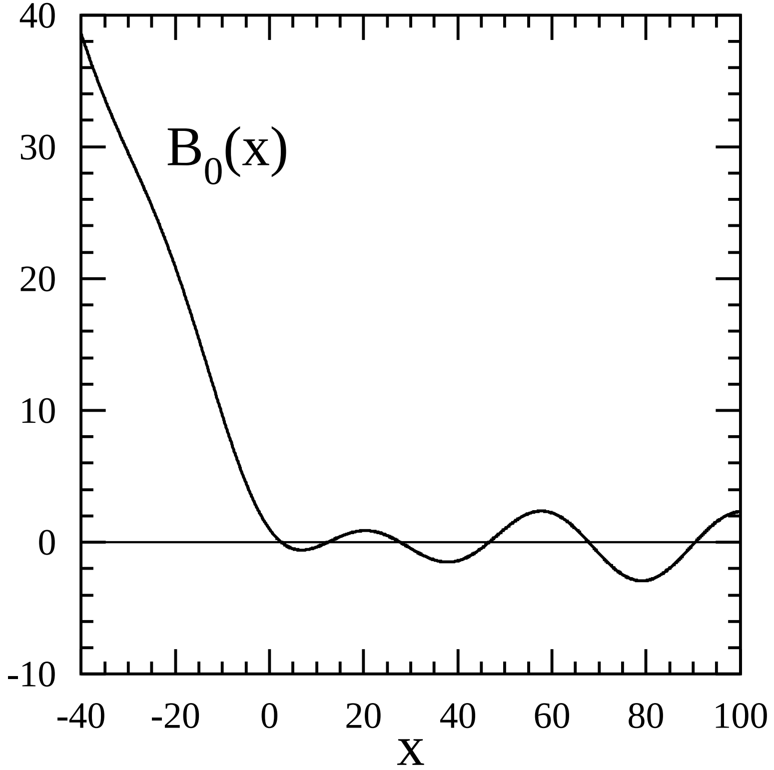

Further $\mathcal{B}$-functions for later use

$$\mathcal{B}_k(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!}\,x^n$$

$$\mathcal{B}_{-k}(x) = \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!}\,x^n$$

Relations to $\mathcal{B}_0(x)$

$$\frac{d^k}{dx^k}(x^k\mathcal{B}_k) = \mathcal{B}_0\,, \quad x^k\frac{d^k}{dx^k}\mathcal{B}_0 = \mathcal{B}_{-k}$$

Next-to-leading logarithmic iteration for $T_{\phi,\mathrm{q}}^{(n)}$

Ansatz for $T_{\phi,\mathrm{q}}^{(n)}$ in terms of first-order quantity and diagonal amplitudes

$$T_{\phi,\mathrm{q}}^{(n)} \overset{\mathrm{NL}}{=} \frac{1}{n} T_{\phi,\mathrm{q}}^{(1)} \left\{ \sum_{i=0}^{n-1} T_{\phi,\mathrm{g}}^{(i)} T_{2,\mathrm{q}}^{(n-i-1)} f(n,i) - \frac{\beta_0}{\varepsilon} \sum_{i=0}^{n-2} T_{\phi,\mathrm{g}}^{(i)} T_{2,\mathrm{q}}^{(n-i-2)} g(n,i) \right\}$$

All-order agreement with known highest four powers of ε^{-1} for

$$\begin{aligned} f(n,i) &= \binom{n-1}{i}^{-1} \left[1 + \varepsilon \left(\frac{\beta_0}{8C_A} (i+1)(n-i)\,\theta_{i1} - \frac{3}{2} (1 - n\,\delta_{i0}) \right) \right] \\ g(n,i) &= \binom{n}{i+1}^{-1} \end{aligned}$$

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$$g(n,i) = \binom{n}{i+1}^{-1}$$

Soft-gluon exponentiation: also $T_{\phi,\mathrm{g}}^{(n)}$ and $T_{2,\mathrm{q}}^{(n)}$ known at all powers of ε

$\Rightarrow$ next-to-leading logarithmic expression for $T_{\phi,\mathrm{q}}$ completely predicted

Mass factorization $\Rightarrow$ $P_{\mathrm{gq}}^{\mathrm{NLL}}$, $c_{\phi,\mathrm{q}}^{\mathrm{NLL}}$ to all orders. $P_{\mathrm{qg}}^{\mathrm{NLL}}$, $c_{2,\mathrm{g}}^{\mathrm{NLL}}$ analogous

Extension of this approach to higher logarithmic accuracy problematic

D-dim. structure of unfactorized observables

Maximal phase space for deep-inelastic scattering/semi-incl. annihilation

$\mathbf{NLO:}\quad 2 \to 2\ /\ 1 \to 1+2 \qquad (1-x)^{-1-\varepsilon}\ x^{\cdots} \int_0^1$ **one other variable**

$\mathbf{N^2LO:}\quad 2 \to 3\ /\ 1 \to 1+3 \qquad (1-x)^{-1-2\varepsilon}\ x^{\cdots} \int_0^1$ **four other variables**

$\mathbf{N^3LO:}\quad 2 \to 4\ /\ 1 \to 1+4 \qquad (1-x)^{-1-3\varepsilon}\ x^{\cdots} \int_0^1$ **seven other variables**

$\ldots$

N^{2}LO: Matsuura, van Neerven (88), Rijken, vN (95), N$^{n\geq 3}$LO, indirectly: MV[V] (05)

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$\mathrm{N^2LO}: \; 2 \to 3 \;/\; 1 \to 1+3$ $\qquad (1-x)^{-1-2\varepsilon} \; x^{\cdots} \int_0^1$ four other variables

$\mathrm{N^3LO}: \; 2 \to 4 \;/\; 1 \to 1+4$ $\qquad (1-x)^{-1-3\varepsilon} \; x^{\cdots} \int_0^1$ seven other variables

...

N^{2}LO: Matsuura, van Neerven (88), Rijken, vN (95), N$^{n\geq 3}$LO, indirectly: MV[V] (05)

Purely real contributions to unfactorized structure functions

$$T_{a,j}^{(n)\mathrm{R}} = (1-x)^{-1-n\varepsilon} \sum_{\xi=0} (1-x)^{\xi} \, \frac{1}{\varepsilon^{2n-1}} \left\{ R_{a,j,\xi}^{(n)\mathrm{LL}} + \varepsilon R_{a,j,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Mixed contributions ($2 \to r+1$ with $n-r$ loops in DIS)

$$T_{a,j}^{(n)\mathrm{M}} = \sum_{l=r}^{n} (1-x)^{-1-l\varepsilon} \sum_{\xi=0} (1-x)^{\xi} \, \frac{1}{\varepsilon^{2n-1}} \left\{ M_{a,j,l,\xi}^{(n)\mathrm{LL}} + \varepsilon M_{a,j,l,\xi}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Purely virtual part (diagonal cases, $\xi = 0$ present): $\gamma^* qq$, Hgg form factors

$$T_{a,j}^{(n)\mathrm{V}} = \delta(1-x) \, \frac{1}{\varepsilon^{2n}} \left\{ V_{a,j}^{(n)\mathrm{LL}} + \varepsilon V_{a,j}^{(n)\mathrm{NLL}} + \ldots \right\}$$

Resulting resummation of large-x double logs

KLN cancellation between purely real, mixed and purely virtual contributions

$$\boldsymbol{T_{a,j}^{(n)} = T_{a,j}^{(n)\mathrm{R}} + T_{a,j}^{(n)\mathrm{M}} \Big(+ T_{a,j}^{(n)\mathrm{V}} \Big) = \frac{1}{\varepsilon^n} \Big\{ T_{a,j}^{(n)0} + \varepsilon T_{a,j}^{(n)1} + \ldots \Big\}}$$

$\Rightarrow$ **Up to $n-1$ relations between the coeff's of $(1-x)^{-1-l\varepsilon}$, $l = 1, \ldots, n$**

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Log expansion: N^{k}LL higher-order coefficients completely fixed, if first $k+1$ powers of ε known to all orders – provided by N^{k}LO calculation, see above

Present situation: (a) N^{3}LO for non-singlet $F_{a \neq L}$ in DIS – recall DMS (05)
(b) N^{2}LO for SIA, non-singlet F_L in DIS, and singlet DIS

$\Rightarrow$ **resummation of the (a) four and (b) three highest $N^{-1} \ln^k N$ terms to all orders in α_s: consistent with, and extending, the results above**

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Soft-gluon exponentiation of the $(1-x)^{-1}/N^0$ diagonal coefficient functions:

$(1-x)^{-1-\varepsilon}, \ldots, (1-x)^{-1-(n-1)\varepsilon}$ at order n: products of lower-order quantities

$\Rightarrow$ **N^{n}LO [+$A^{(n+1)}$] $\rightarrow$ N^{n}LL exponentiation; $2n[+1]$ highest logs predicted**

Selection of some new results

NS cases: $K_{a,4}(x)$ **of p.7 confirmed with** $\xi_{K_4} = 100/3$**: fourth log for** $c_{a,\mathrm{ns}}^{(n\geq 4)}$

Off-diagonal splitting functions

$$\tilde{a}_{\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{\pi}\,(C_A - C_F)\,\ln^2 N$$

$$\begin{aligned}
NP_{\mathrm{qg}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) \;&=\; 2a_{\mathrm{s}}\, n_f\, \mathcal{B}_0(\tilde{a}_{\mathrm{s}}) \\
&\quad + a_{\mathrm{s}}^2 \ln N\, n_f \left\{ (6C_F - \beta_0)\left(\frac{2}{\tilde{a}_{\mathrm{s}}}\mathcal{B}_{-1}(\tilde{a}_{\mathrm{s}}) + \mathcal{B}_1(\tilde{a}_{\mathrm{s}})\right) + \frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\,\mathcal{B}_{-2}(\tilde{a}_{\mathrm{s}}) \right\}
\end{aligned}$$

$$\begin{aligned}
NP_{\mathrm{gq}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) \;&=\; 2a_{\mathrm{s}}\, C_F\, \mathcal{B}_0(-\tilde{a}_{\mathrm{s}}) \;+\; a_{\mathrm{s}}^2 \ln N\, C_F \left\{ (12C_F - 6\beta_0)\,\frac{1}{\tilde{a}_{\mathrm{s}}}\,\mathcal{B}_{-1}(-\tilde{a}_{\mathrm{s}}) \right. \\
&\quad \left. - \frac{\beta_0}{\tilde{a}_{\mathrm{s}}}\,\mathcal{B}_{-2}(-\tilde{a}_{\mathrm{s}}) + \;(14C_F - 8C_A - \beta_0)\,\mathcal{B}_1(-\tilde{a}_{\mathrm{s}}) \right\}
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Gluon contribution to F_L – 'non-singlet' $C_F = 0$ part done before MV (09)

$$
\begin{aligned}
N^2 c_{L,\mathrm{g}}^{\mathrm{NL}}(N,\alpha_{\mathrm{s}}) \;&=\; 8a_{\mathrm{s}}n_f\,\exp(2C_A a_{\mathrm{s}}\ln^2 N) \;+\; 4a_{\mathrm{s}}C_F\,N C_{2,\mathrm{g}}^{\mathrm{LL}}(N,\alpha_{\mathrm{s}}) \\
&\quad + 16a_{\mathrm{s}}^2 \ln N\, n_f\left\{4C_A - C_F + \frac{1}{3}a_{\mathrm{s}}\ln^2 N\, C_A\beta_0\right\}\exp(2C_A a_{\mathrm{s}}\ln^2 N)
\end{aligned}
$$

NNLL contributions known to 'any' order, but (mostly) no closed expressions

Summary and outlook

- Non-singlet physical kernels for nine observables in DIS, SIA and DY:
  single-log large-x enhancement at NNLO$/$N^{3}LO to all orders in $1-x$
  All-order conjecture $\Rightarrow$ leading three (DY: two) logs of higher-order C_a
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  $\Rightarrow$ Prediction of three logs in N^{3}LO α_s^4 splitting and F_L coefficient fct's

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- Complementary approach: Laenen, Magnea, Stavenga, White (from 08)
- Limited phenomenol. relevance now: assess relevance of NS $1/N$ terms
- Near$/$mid future: combine with other results, esp. fixed-N calculations
  (close to) feasible now: 4-loop sum rules Baikov, Chetyrkin, Kühn (10)