The *a*-theorem in various dimensions

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Outline

Introduction to the a-theorem

The a-theorem in $D=4$

The a-theorem in $D=6$ - *with J. A. Gracey*

The a-theorem in $D=3$ - *with D. R. T. Jones*

Introduction

Consider a renormalizable QFT with couplings g_I at energy scale μ. Zamolodchikov (1986) showed:

- For 2D QFT, $\exists\, c(\mu, g_I)$ decreasing monotonically under RG flow.
- At a fixed point g_I^*, $c(\mu, g_I^*) = c$, the central charge.

The central charge of a 2D QFT appears in the Trace anomaly:

The Trace anomaly

$$
\begin{aligned}
2D : \langle T^\mu_{\ \mu} \rangle &= -\frac{c}{12} R \\
4D : \langle T^\mu_{\ \mu} \rangle &= cF - \frac{1}{4} \mathbf{a} G + \ldots
\end{aligned}
$$

Cardy (1988) conjectured there could be a generalization:

- For **4D** QFT, $\exists\,\mathbf{A}(\mu, g_I)$ decreasing monotonically under RG flow. (*strong* a-theorem)
- At a fixed point g_I^*, $A(\mu, g_I^*) = \frac{1}{4}a$.
- Alternatively, $\exists\,A(\mu, g_I)$ such that $a_{UV} - a_{IR} > 0$. (*weak* a-theorem)

Jack and Osborn (1990) subsequently showed that

- For 4D QFT, $\exists\,A$ such that $\partial_I A = T_{IJ}\beta^J$.
- At a fixed point g_I^*, $A(\mu, g_I^*) = \frac{1}{4}a$.

Hence, if $G_{IJ} = T_{(IJ)}$ is positive-definite, then

$$\mu\frac{d}{d\mu}A = \beta^I\partial_I A = \beta^I G_{IJ}\beta^J;$$

that is, A monotonically decreases along RG flow.

The key equation

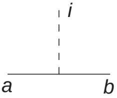
$$dA = dg^I T_{IJ} \beta^J \tag{1}$$

Even if G_{IJ} is not positive-definite, the existence of such an A provides consistency conditions on the form of the β-functions in a QFT.

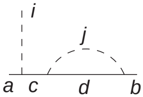
So, we can expand perturbatively and solve!

The *a*-theorem in *D*=4

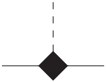
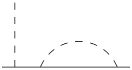
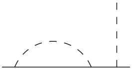
Published in ***JHEP* 1501 (2015) 138**.
General theory has couplings $\{\lambda, y, g\}$. We use diagrammatic notation, e.g. Yukawa coupling:

$y_{iab} \rightarrow$ 

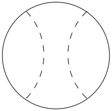
Summed indices represent contractions:

$y_{iac}y_{jcd}y_{jdb} \rightarrow$ 

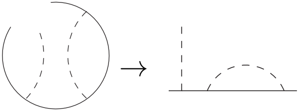
Represent β-function as sum of diagrams with coefficients:

$\beta^{(y)}_{iab} \rightarrow$  $= b_1 \Big($  $+$  $\Big) + \ldots$

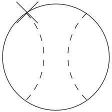
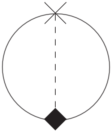
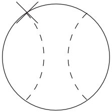
Terms in the A function are represented by completely contracted diagrams:

$A = A_1$  $+ \ldots$

Removing a vertex gives a β-function contribution:



So, by equation (1), and denoting differentiation by a cross on a vertex:

$4A_1$  $+ \ldots \quad = \quad \alpha$  $\quad = \quad 2b_1\alpha$  $+ \ldots$

Hence, $A_1 = \frac{\alpha}{2} b_1$, and so on for each term in $\beta^{(y)}$. By explicit calculation, $\alpha = \frac{1}{6}$.

Extending to next order, the equations to solve are:

$$d_y A^{(4)} = dy T^{(2)}_{y\bar{y}} \beta^{(2)}_{(y)} + dy T^{(3)}_{y\bar{y}} \beta^{(1)}_{(y)}$$

$$d_\lambda A^{(4)} = d\lambda T^{(3)}_{\lambda\lambda^*} \beta^{(1)}_{(\lambda)}$$

Yukawa differential equation is non-trivial: 6 consistency conditions on the 2-loop Yukawa β-function.

Expect conditions to be scheme-independent; can verify by coupling redefinition:

$$\delta\beta^{(2)}_I = \left(\beta^{(1)}_J \frac{\delta}{\delta g_J}\right) \delta g_I - \left(\delta g_J \frac{\delta}{\delta g_J}\right) \beta^{(1)}_I$$

Bonus features:

- Prediction of general 3-loop gauge β-function.
- Reduces to $\mathcal{N}=1$ Supersymmetric results via correct coupling/field definitions.

Supersymmetric case has a potential all-orders expression for A, will be satisfied if the following equation holds:

The Λ equation

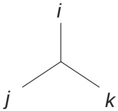
$$3\bar{y}\cdot\Lambda - 2\lambda C_R = \gamma - \gamma^2 + \Theta \circ \beta_{\bar{y}} + \theta\tilde{\beta}_g$$

Similar loop-by-loop calculation with general anomalous dimension γ gives consistency conditions on $\gamma^{(3)}$.

The *a*-theorem in D=6

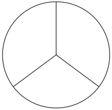
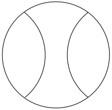
Since the Trace anomaly exists in all even dimensions, there may exist an *a*-theorem formulation in all even dimensions. Test with ϕ^3 theory in six dimensions.
Construction of A more straightforward, since there is only one coupling type:

$g_{ijk} \to$



Can still derive constraints.
To test scheme independence, use standard $\overline{MS}$ results, and *MOM* results calculated by J. A. Gracey.

1-loop β-function contains only two terms, hence lowest order A function contains only two terms:

$A^{(3)} = A_1^{(3)}$  $+ A_2^{(3)}$ 

However... lowest order metric is *negative*-definite: $\lambda = -\frac{1}{3240}$.
In lower dimensions A should count "degrees of freedom"...
What is happening here?

Next order calculation including 2-loop β-function produces one consistency condition. Can use coupling redefinitions to show scheme independence as in 4D; explicitly verified using $\overline{MS}$ and MOM results.

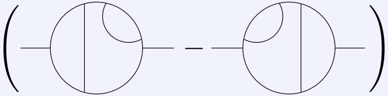
3-loop more interesting...

3-loop gives 8 conditions on coefficients. Multiple constraints include combination $d_{19} + d_{30}$, last constraint has $d_{19} - d_{30}$: not satisfied!

Solution is a contribution to β, introduces shift $\beta_I \to B_I = \beta_I - (Sg)_I$:

A new contribution

$S_{ij} = \theta$



New term introduces a term proportional to θ in consistency condition: satisfied if $\theta = \frac{137}{10368}$. Much easier than 3-loop curved spacetime integrals!

The *a*-theorem in *D*=3

Preprint ***arXiv:1505.05400***.

In $D = 3$, Trace anomaly vanishes (can be seen from Heat Kernel methods and DREG).

Even if Trace anomaly were non-zero, the Euler characteristic (hence Euler density) vanishes in odd dimensions: seems to be no candidate for an A-function...

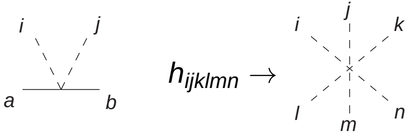
... or is there?

If one takes a $D = 3$ theory, e.g. Chern-Simons, $\exists$ an A-function with positive-definite metric!

In 3D theories, β-functions are 0 at odd loop orders: only even loop order terms exist. (1) expands perturbatively only using even order terms.

Couplings are:

$y_{ijab} \rightarrow$ $h_{ijklmn} \rightarrow$



As before, Lowest order A is trivial to construct.

At next order, can construct terms using $\beta_h^{(2)}$, differentiate with respect to y_{ijab} to obtain predictions for $\beta_y^{(4)}$: predictions are correct.

As in other dimensions, can obtain consistency conditions from next order Yukawa equation.

General results reduce to multiple explicit cases:

- Abelian Chern-Simons
- Non-Abelian $SU(N)$ Chern-Simons
- Non-Abelian $\mathcal{N} = 1$ $SO(N)$ Chern-Simons

This is encouraging, but to satisfy the criteria of the A-function, need to know A at fixed points. Even more crucially... *what is A!?*

Conclusions

4D case: holding up perturbatively for general QFTs.

- Non-trivial consistency conditions between β-function coefficients
- Determination of higher-loop β-function coefficients from lower orders
- All-order supersymmetric condition satisfied up to three loops

6D case: still useful even though metric negative-definite?

- Non-trivial consistency conditions explicitly verified in two schemes
- Contribution to $\langle T^{\mu}_{\mu} \rangle$ at 3 loops calculated

3D case: Surprisingly able to construct A-function

- ... let's go with "more work needed".