AN a-THEOREM IN THREE DIMENSIONS?

with Tim Jones and Colin Poole

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The c and a theorems

Consider a theory with couplings $\{g^I\}$. The β-functions are defined by

$$\beta^I = \mu \frac{d}{d\mu} g^I$$

β-functions describe how couplings change with energy scale; derived from simple poles in counterterms (using dimensional regularisation with $d = 2 - \epsilon, 4 - \epsilon, \ldots$).

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β-functions describe how couplings change with energy scale; derived from simple poles in counterterms (using dimensional regularisation with $d = 2 - \epsilon, 4 - \epsilon, \ldots$).
The Zamolodchikov c-theorem (two dimensions) says that there is a function c such that

$$\partial_I c = G_{IJ}\beta^J(g) \quad \text{or} \quad d_g c \equiv dg^I \partial_I c = dg^I G_{IJ}\beta^J,$$

so that

$$\mu \frac{d}{d\mu} c = \beta^I \partial_I c = G_{IJ}\beta^I \beta^J,$$

where G_{IJ} is a positive definite metric.

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- The strong a-theorem: There is a function $a(g)$ which has monotonic behaviour under RG flow. Gradient flow proved except for monotonicity. Positive definiteness of G_{IJ} easily checked to lowest order$\Rightarrow$ valid perturbatively to all orders.

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In three dimensions there is no trace anomaly and therefore no candidate for an a-function.
Nevertheless we can show that 3-d theories do satisfy a "gradient flow" equation with a positive definite metric at lowest order.

Trace anomaly

Theory with generic fields φ: Conformal invariance: invariance under local Weyl rescalings

$$\gamma_{\mu\nu} \to \Omega^2(x)\gamma_{\mu\nu}, \quad \varphi \to \Omega^{2p}(x)\varphi.$$

To leading order we find

$$\delta S[\varphi, \gamma_{\mu\nu}] = \int d^d x \left(\frac{\delta S}{\delta \varphi} \delta\varphi + \frac{\delta S}{\delta \gamma_{\mu\nu}} \delta\gamma_{\mu\nu} \right) = 0.$$

Using Lagrange equations $\frac{\delta S}{\delta \varphi} = 0$ we have

$$\gamma_{\mu\nu} \frac{\delta S}{\delta \gamma_{\mu\nu}} = -\gamma_{\mu\nu} T^{\mu\nu} = -T^{\mu}{}_{\mu} = 0, \quad T^{\mu\nu} = -\frac{\delta S}{\delta \gamma_{\mu\nu}}.$$

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For a scalar field ϕ,

$$L = \tfrac{1}{2}|\gamma|^{\frac{1}{2}}(\gamma^{\mu\nu}\partial_\mu\phi\partial^\nu\phi - m^2\phi^2 - \xi R\phi^2)$$

need to take $m^2 = 0$, $p = \frac{2-d}{4}$, $\xi = \frac{d-2}{4(d-1)}$.

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- In general $T^{\mu\nu}$, $\Theta^{\mu\nu}$ differ by total derivative.
- $T^{\mu\nu}$ gives symmetric result (required by Einstein equation)
- $\Theta^{\mu\nu}$ gives closer relation to “energy” and “momentum”.

In quantum theory define

$$< T_{\mu\nu} >= \frac{\partial}{\partial \gamma^{\mu\nu}} W, \quad W = -i \ln Z, \quad Z = \int d[\phi] e^{iS[\phi]}.$$

Trace anomaly

Expect scale dependence to be introduced at quantum level through scale dependence of couplings so

$$< T^{\mu}{}_{\mu} >= -\gamma^{\mu\nu}\frac{\partial}{\partial\gamma_{\mu\nu}}W = \beta_I\frac{\partial}{\partial g^I}W \equiv \beta_I\mathcal{O}^I.$$

However on curved space need additional counterterms

$$c_B R \to \Omega^{-2} c_B R + \ldots, \quad d^d x|\gamma|^{\frac{1}{2}} \to \Omega^d d^d x|\gamma|^{\frac{1}{2}}$$

(in two dimensions) and

$$\begin{aligned} &A_B G + B_B F + \tilde{C}_B R^2, \\ &\quad F = C^{\mu\nu\sigma\rho} C_{\mu\nu\sigma\rho} \to \Omega^{-4} F, \\ &\quad G = \frac{1}{4}\epsilon_{\mu\nu\sigma\rho}\epsilon^{\alpha\beta\gamma\delta} R^{\mu\nu}{}_{\alpha\beta} R^{\sigma\rho}{}_{\gamma\delta} \to \Omega^{-4} G + \ldots \end{aligned}$$

(in four dimensions) ($C_{\mu\nu\sigma\rho}$ is the conformal tensor.)

Trace anomaly

These lead to additional contributions, so

$$
\begin{aligned}
< T^{\mu}{}_{\mu} > &= \beta_I \mathcal{O}^I + cR + \ldots \quad (2-d) \\
&= \beta_I \mathcal{O}^I + aG + bF + \tilde{c}R^2 + \ldots \quad (4-d),
\end{aligned}
$$

where $c, a, b, \tilde{c}$ are the β-functions corresponding to $c_B, A_B, B_B, \tilde{C}_B$.

Derivation of c-theorem

Extend to x dependent couplings $g^I(x)$ and consider infinitesimal Weyl rescalings

$$\Omega = 1 + \sigma(x), \quad \delta_\sigma \gamma_{\mu\nu} = 2\sigma\, \gamma_{\mu\nu}$$

which are implemented by the operator

$$\Delta_\sigma = \int d^2x\, \sigma \left(2\gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + \beta^I \frac{\delta}{\delta g^I} \right)$$

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Acting on the vacuum energy functional $W[\gamma_{\mu\nu}, g^I]$, we get

$$\Delta_\sigma W = - \int d^2x \sqrt{-\gamma}\, \sigma \left(cR + G_{IJ} \gamma^{\mu\nu} \partial_\mu g^I \partial_\nu g^J \right)$$
$$- 2 \int d^2x \sqrt{-\gamma}\, \gamma^{\mu\nu} \partial_\mu \sigma \;\; W_I \partial_\nu g^I$$

Derivation of *c*-theorem

Derivation of c-theorem

Group of local Weyl transformations is abelian:

$$\left[\Delta_\sigma, \Delta_{\sigma'}\right] = 0$$

Using

$$\delta_\sigma R = -2\sigma R - 2\nabla^2\sigma \quad \text{etc}$$

implies consistency conditions

$$\partial_I \tilde{c} = T_{IJ}\beta^J(g)$$
$$\tilde{c} = c + W_I\beta^I, \quad T_{IJ} = G_{IJ} + \partial_I W_J - \partial_J W_I$$

Chern-Simons theory

Chern-Simons theory: gauge theory in three dimensions

- Fractional statistics
- Quantum Hall effect
- High T_c superconductivity
- AdS/CFT corrrespondence: superconformal $\mathcal{N}=2$ theories, e.g. BLG, ABJ models

Pure non-abelian Chern-Simons theory:

$$L = \frac{1}{2}\epsilon^{\mu\nu\rho}A^a_\mu\partial_\nu A^a_\rho + \frac{1}{6}ef^{abc}\epsilon^{\mu\nu\rho}A^a_\mu A^b_\nu A^c_\rho$$

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- Pure Chern-Simons theory (no matter) is topological and the gauge coupling is quantised
- With matter: no infinite counterterms for e and no β-function–the gauge coupling does not run.
- However there are finite corrections to e

An explicit example

We start with a particular case of the abelian Chern-Simons theory with matter:

$$L = \tfrac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + |D_\mu\phi_j|^2 + i\overline{\psi}_j\hat{D}\psi_j + \alpha\overline{\psi}_j\psi_j\phi_k^*\phi_k$$
$$+ \beta\overline{\psi}_j\psi_k\phi_k^*\phi_j + \tfrac{1}{4}\gamma(\overline{\psi}_j\psi_k^*\phi_j\phi_k + \overline{\psi}_j^*\psi_k\phi_j^*\phi_k^*) - h(\phi_j^*\phi_j)^3.$$

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The two-loop β-functions for this theory are given by

$$\beta_\alpha^{(2)} = \left(\tfrac{8}{3}n + 2\right)\alpha^3 + \tfrac{16}{3}\alpha^2\beta + \left(\tfrac{8}{3}n + 3\right)\alpha\beta^2 + (n+2)\beta^3$$
$$+ \tfrac{1}{4}\left(\tfrac{8}{3}n + \tfrac{17}{3}\right)\alpha\gamma^2 + \tfrac{3}{4}(n+2)\beta\gamma^2 + 3\beta^2e^2 + \tfrac{1}{4}\gamma^2e^2$$
$$- \tfrac{2}{3}(20n+31)\alpha e^4 - 8\beta e^4 - 8(n+2)e^6,$$
$$\beta_\beta^{(2)} = \left(\tfrac{8}{3}n + 6\right)\alpha^2\beta + \left(3n + \tfrac{16}{3}\right)\alpha\beta^2 + \left(\tfrac{2}{3}n + 1\right)\beta^3$$
$$+ \tfrac{3}{4}(n+2)\alpha\gamma^2 + \tfrac{1}{4}\left(\tfrac{8}{3}n + \tfrac{17}{3}\right)\beta\gamma^2 - 3n\beta^2e^2$$
$$+ \tfrac{1}{4}(n+2)\gamma^2e^2 - \tfrac{2}{3}(8n+31)\beta e^4,$$

An explicit example

$$\beta_\gamma^{(2)} = \left(\tfrac{8}{3}n + 6\right)\alpha^2\gamma + \left(6n + \tfrac{34}{3}\right)\alpha\beta\gamma + \left(\tfrac{8}{3}n + 6\right)\beta^2\gamma$$
$$+ \tfrac{1}{6}(n+1)\gamma^3 + 4\alpha\gamma e^2$$
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It is straightforward to show that the β-functions satisfy

$$\begin{pmatrix} \partial_\alpha A \\ \partial_\beta A \\ \partial_\gamma A \end{pmatrix} = \begin{pmatrix} n & 1 & 0 \\ 1 & n & 0 \\ 0 & 0 & \frac{1}{4}(n+1) \end{pmatrix} \begin{pmatrix} \beta_\alpha^{(2)} \\ \beta_\beta^{(2)} \\ \beta_\gamma^{(2)} \end{pmatrix},$$

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(Note that the metric is positive definite!)

An explicit example

where

$$A = \frac{n}{4}\left(\frac{8}{3}n+2\right)\alpha^4 + \frac{1}{6}\left(n^2+3n+3\right)\beta^4 + \frac{1}{96}(n+1)^2\gamma^4$$
$$+\left(\frac{8}{3}n+2\right)\alpha^3\beta + \frac{1}{3}(3n^2+8n+3)\beta^3\alpha + (1-n^2)\beta^3 e^2$$
$$+\frac{1}{3}(4n^2+9n+8)\alpha^2\beta^2 + \frac{1}{12}(4n+9)(n+1)(\alpha^2+\beta^2)\gamma^2$$
$$+\frac{1}{12}(9n^2+26n+17)\alpha\beta\gamma^2 + \frac{1}{2}(n+1)\alpha\gamma^2 e^2 + \frac{1}{4}(n+1)^2\beta\gamma^2 e^2$$
$$-\frac{n}{3}(20n+31)\alpha^2 e^4 - \frac{1}{3}(8n^2+31n+12)\beta^2 e^4$$
$$-\frac{n}{3}(2n-5)\gamma^2 e^4 - \frac{2}{3}(20n+31)\alpha\beta e^4 - 8n(n+2)\alpha e^6.$$

General case

Use a general lagrangian

$$L = \frac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + |D_\mu\phi_j|^2 + i\overline{\psi}_j\hat{D}\psi_j + \alpha_{ijkl}\overline{\psi}_j\psi_j\phi^*_k\phi_l - h_{ijklmn}\phi^i\phi^j\phi^k\phi^l\phi^m\phi^n.$$

We can construct $A^{(5)}$, $A^{(7)}$ such that

$$d_\alpha A^{(5)} = d\alpha^{ijkl}\beta_\alpha^{ijkl(2)} \quad d_h A^{(7)} = dh^{ijklmn}\beta_h^{ijklmn(2)}$$

or schematically

$$d_\alpha A^{(5)} = d\alpha^I g_{IJ}\beta_\alpha^{J(2)}, \quad g_{IJ} = \delta_{IJ},$$
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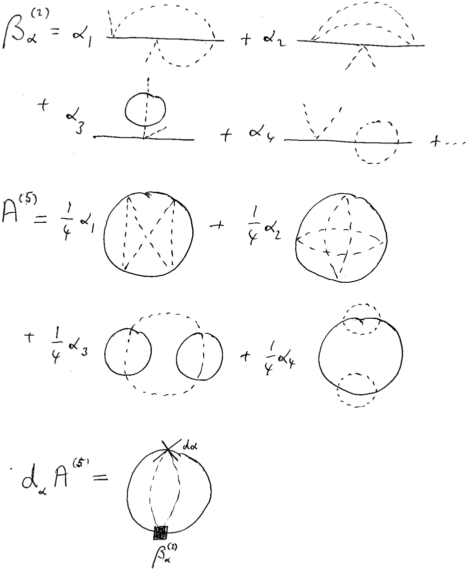
whatever the coefficients in $\beta^{(2)}$! h-dependent terms in $A^{(7)}$ lead to predictions for 4-loop h-dependent contributions to β^α!

General case



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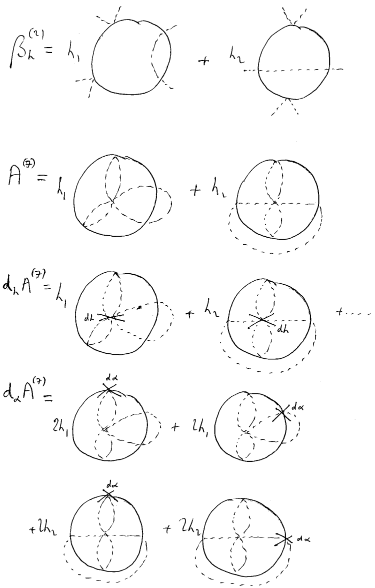
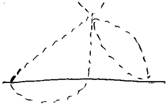
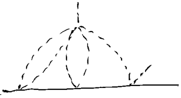
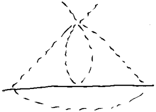
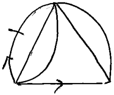
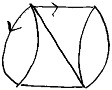


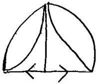
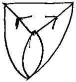
General case

S_0:

$\beta_\alpha^{(4)} = 2h_1$  $+ 2h_1$ 

$+ 2h_2$  $+ 2h_2$ 

$\Rightarrow$  $=$ 

$= \frac{1}{2}$  $= \frac{1}{2}$ 

General case

At higher orders one expects (if the gradient flow continues to hold)

$$d_\alpha A^{(7)} = d\alpha^I \beta_\alpha^{I(4)} + d\alpha^I T_{IJ}^{(5)} \beta_\alpha^{I(2)}$$

–the leading order term determined by earlier calculation and $T_{IJ}^{(5)}$ is unknown.

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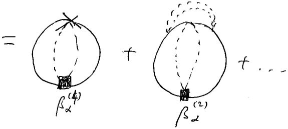
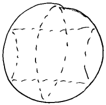
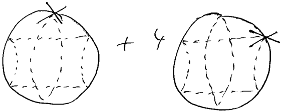
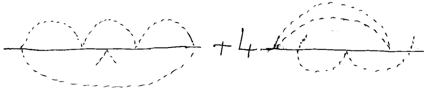


If $A^{(7)} \supset$



$dA^{(7)} \supset 2$ 

$\beta_\alpha^{(4)} \supset 2$ 

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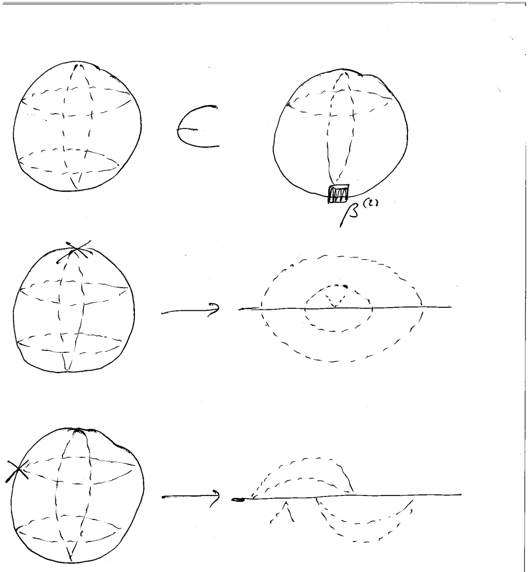

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