Explorations in entanglement and black holes

University of Liverpool 25/02/2015

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EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

NYT headline, May 4, 1935

What is entanglement entropy?

Consider a two-particle system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\Psi\rangle = \sum_{i,j} a_{ij} |i\rangle \otimes |j\rangle$$

If we can factorize it the state it isn't entangled

Example: Spin up $|\uparrow\rangle$ Spin down $|\downarrow\rangle$

Which state is entangled?

$$\begin{aligned} |\Psi_1\rangle &= \frac{1}{\mathcal{N}} (|\uparrow\rangle|\downarrow\rangle + |\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \\ |\Psi_2\rangle &= \frac{1}{\mathcal{N}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \end{aligned}$$

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$|\Psi_2\rangle$ is and Einstein-Podolsky-Rosen (EPR) pair

Diagnosing entanglement

Step 1: $|\Psi\rangle \longrightarrow$ reduced density matrix

$$\rho_A = \mathrm{Tr}_B \left(|\Psi\rangle\langle\Psi| \right)$$

Step 2: Density matrix $\longleftrightarrow$ Von Neumann entropy

$$S_{EE} = -\mathrm{Tr} \left(\rho_A \log \rho_A \right)$$

Step 3: Is $S_{EE} = 0$ or $S_{EE} \neq 0$?

Example: Two-spin system

$$\rho_A = \begin{pmatrix} |a_{\downarrow\downarrow}|^2 + |a_{\downarrow\uparrow}|^2 & \bar{a}_{\uparrow\downarrow} a_{\downarrow\downarrow} + \bar{a}_{\uparrow\uparrow} a_{\downarrow\uparrow} \\ a_{\uparrow\downarrow} \bar{a}_{\downarrow\downarrow} + a_{\uparrow\uparrow} \bar{a}_{\downarrow\uparrow} & |a_{\uparrow\uparrow}|^2 + |a_{\uparrow\downarrow}|^2 \end{pmatrix}$$

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Example: Two-spin system $|\Psi_1\rangle$

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad S_{EE} = 0$$

Diagnosing entanglement

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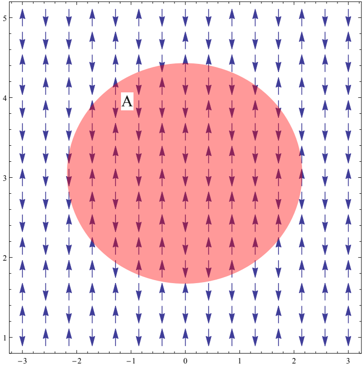
Example: Two-spin system $|\Psi_2\rangle$

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_{EE} = \log(2)$$

Entanglement in field theory

Lattice system $\mathcal{H} = \bigotimes_i \mathcal{H}_i$



Ground state (or low-lying)

$$S_{EE}(A) \sim \#\text{ links cut by } \partial A + \ldots$$
$$\sim \left(\frac{L}{\epsilon}\right)^{d-2} + \ldots$$
$$\sim \log\frac{L}{\epsilon} + \ldots \quad (\text{in } d = 2\ )$$

Bombelli, Koul, Lee, ... '86

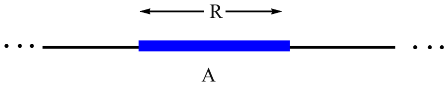
Important

$$\epsilon \to 0 \implies S_{EE}(A) \to \infty$$

It is possible to remove the UV singularities consistently in the $\epsilon \to 0$ limit

Renormalized entanglement entropy

Example: critical models $d = 2$



$$S(A) = \frac{c}{3} \log\left(\frac{R}{\epsilon}\right) + \dots$$

Calabrese, Cardy, '04

In general, it is possible to extract the universal contributions to EE

$$d = 2 \quad : \qquad \mathcal{S}(R) = R\frac{d}{dR} S_{EE}(A)$$

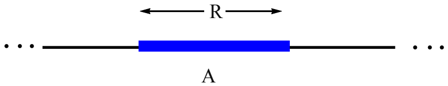
$$d = 3 \quad : \qquad \mathcal{S}(R) = \left(R\frac{d}{dR} - 1\right) S_{EE}(A)$$

$$d = 4 \quad : \qquad \mathcal{S}(R) = \frac{1}{2} R\frac{d}{dR}\left(R\frac{d}{dR} - 2\right) S_{EE}(A)$$

Casini, Huerta, '04 Liu, Mezei '12

Renormalized entanglement entropy

Generically, for $d = 2n$



$$S(A) = \ldots a_d^* \log\left(\frac{R}{\epsilon}\right) + \ldots$$

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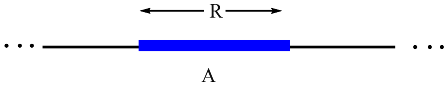
Casini, Huerta, '04 Liu, Mezei '12

Renormalized entanglement entropy

Generically, for $d = 2n + 1$



$$S(A) = \ldots a_d^* + \ldots$$

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Casini, Huerta, '04 Liu, Mezei '12

Renormalized entanglement entropy

Entangling region S^{d-1}

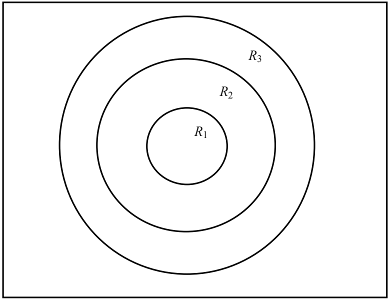
In a CFT $\mathcal{S} = \text{const}$

Under RG-flow

$$\mathcal{S}(R_1) > \mathcal{S}(R_2) > \mathcal{S}(R_3)$$

It provides a "c-function"!!

$$\partial_R \mathcal{S} < 0$$



Casini, Huerta, '04 Liu, Mezei '12

Relates naturally to QFT monotonicity theorems: the c-theorem (Zamolodchikov '86), a-theorem (Schwimmer, Komargodski '11) and F-theorem (Klebanov etal '13) .

The Ryu-Takayanagi prescription

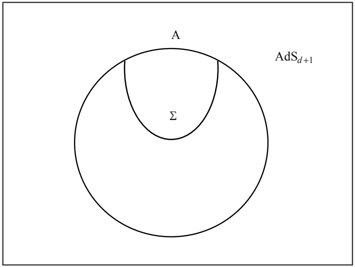
Field Theory dual to EH

State $\leftrightarrow$ static space-time

$$S_{EE} = \frac{1}{4G} \min_{\Sigma} \text{Area}(\Sigma)$$

Σ co-dimension 2, $\partial\Sigma = \partial A$

Ryu, Takayanagi '06



The RT formula passes many tests

1. Reproduces replica computations, Lewkowycz, Maldacena '13
2. Obeys basic properties of EE, e.g SSA,... Headrick, Takayanagi '07
3. BTZ yields Calabrese-Cardy Ryu, Takayanagi '06
4. Covariant version Hubeny, Rangamani, Takayanagi '07

REE for BPS black objects

A. Bhattacharrya, S. Haque and AV-O '14

$\mathcal{N} = 2$ gSUGRA in a nutshell

Lagrangian in 4d and 5d

$$\mathcal{L} = \frac{1}{2} R - G_{i\bar{\jmath}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^j + \frac{1}{4} \Im \mathcal{N}_{IJ} F^I_{\mu\nu} F^{J\,\mu\nu} - g^2 V_F(\phi, \bar{\phi})$$

- The field content is:
  - Dynamical metric $g_{\mu\nu}$ two derivative EH
  - Complex scalar fields (Moduli) $\phi^i \quad i = 1, \ldots, n_v$
  - Abelian gauge fields $A^I_\mu \quad I = 1, \ldots, n_v + 1$
- Structure determined by holomorphic prepotential:
  - $G_{i\bar{\jmath}}$ is a special Kähler metric.
  - $\mathcal{N}_{IJ}$ scalar dependent gauge couplings.
  - $V_F(\phi, \bar{\phi})$ flux potential.

$\mathcal{N} = 2$ gSUGRA in a nutshell

Period vector and symplectic transformations

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \qquad \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}$$

Prepotential and structure constants

$$F = -c_{ijk} \frac{X^i X^j X^k}{X^0} \equiv -c_{ijk} \left(X^0\right)^2 \phi^i \phi^j \phi^k$$

Charges and FI gaugings

$$\Gamma = \begin{pmatrix} P^I \\ Q_I \end{pmatrix} \qquad\qquad G = \begin{pmatrix} h^I \\ h_I \end{pmatrix}$$

Solutions with symplectically related vectors are physically equivalent

$\frac{1}{4}$-BPS black objects

AdS Black solutions $\longleftrightarrow$ Purely algebraic problem

$$ds^2 = -a(r)^2 dt^2 + a(r)^{-2} dr^2 + b(r) d\Omega_k \qquad k = 0, \pm 1$$

Everything writen with **symplectic invariant**

$$I_4\left(V^{(1)}, V^{(2)}, V^{(3)}, V^{(4)}\right) = t^{MNPQ}\, V_M^{(1)}\, V_N^{(2)}\, V_P^{(3)}\, V_Q^{(4)}.$$

Entropy	AdS_4 radius	AdS_2 radius
$S_{BH} = \pi\sqrt{I_4(B)}$	$L_4 = I_4(G)^{-1/4}$	$L_2 = \langle G, B\rangle^{-1} I_4(B)^{1/4}$

Halmagyi, Katmadas 2014

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$$a\,b = \Big(I_4(G)^{1/4}\,r + \langle G, B\rangle\Big)\,r \qquad b = \frac{1}{2}\,I_4(H)^{1/4} \qquad H = A\,r + B$$

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$$\frac{1}{4} dI_4(B, B, G) = \Gamma$$

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Halmagyi, Katmadas 2014

Entanglement entropy for BPS black objects

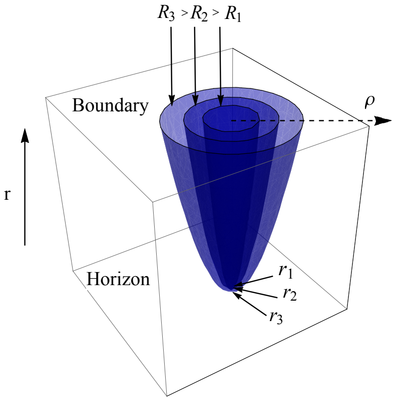
Subsystem A, disk radius R

$$S(R) = \frac{2\pi}{l_p^2} \int_0^R d\rho\, \rho\, b^2 \sqrt{1 + e^{-2\psi} \dot{r}^2}$$

Where $\partial_\rho = \dot{}$ and $e^\psi = ab$

Problem:

For a given depth r_*, $r(\rho) = ??$



Plateau equation: $$\ddot{r} - \psi' \, \dot{r}^2 + \left(\frac{\dot{r}}{\rho} - 2\frac{b'}{b} e^{2\psi} \right) \left(1 + e^{-2\psi} \, \dot{r}^2 \right) = 0$$

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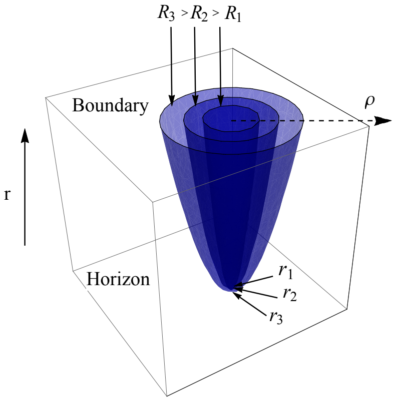

BPS Black holes: It is quite a complicated equation!!

Entanglement entropy for BPS black objects

Using NDSolve with $r(0) = r_*$ and $\dot{r}(0) = 0$

$$\ddot{r} - \psi' \, \dot{r}^2 + \left(\frac{\dot{r}}{\rho} - 2\frac{b'}{b} e^{2\psi} \right) \left(1 + e^{-2\psi} \, \dot{r}^2 \right) = 0$$

Gives an interpolating function, but...

NDSolve::ndsz : At ρ == 0.40753176568989086`, step size is effectively zero; singularity or stiff system suspected. ≫

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What seems a curse is a blessing

Depth into the bulk $\longleftrightarrow$ Entangling disk radius

Very handy to relate EE and RG

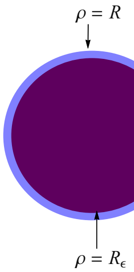
Entanglement entropy for BPS black objects

S_{EE} = Area evaluated on the solution $\tilde{r}(\rho)$

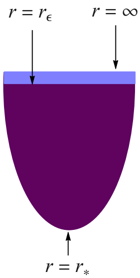
Area needs to be regulated

Bulk $\quad r < r_\epsilon$, $\quad \dfrac{l_4}{b(r_\epsilon)} = \epsilon$

Boundary $\quad \rho < R_\epsilon$, $\quad \dfrac{l_4}{b(\tilde{r}(R_\epsilon))} = \epsilon$



Boundary



Bulk

Construct a function $\quad S^{(\mathrm{reg})}(R,\epsilon) = \displaystyle\int_0^{R_\epsilon} d\rho\, \mathcal{A}(R,\rho)$

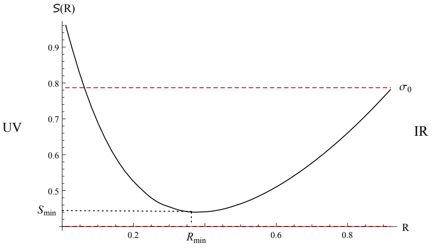
Entanglement entropy for BPS black objects

Extract the renormalized entanglement entropy Independent of ϵ

REE for $d+1=4$

$$\mathcal{S}(r) = \left(R\frac{\partial}{\partial R} - 1\right) S^{(\text{reg})}$$

Pure AdS_4 $\mathcal{S}(r) = 1$



Intuitive field theory picture

Black objects $\longleftrightarrow$ Mixed density matrix

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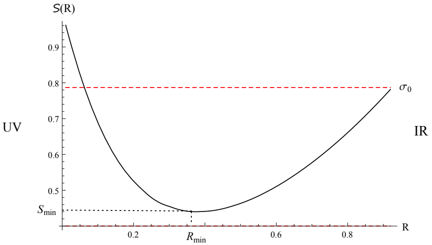
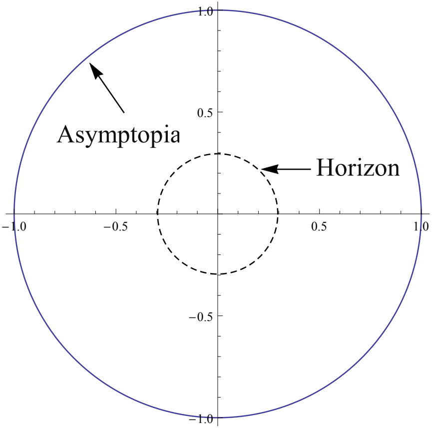


Intuitive field theory picture

$\rho_A \approx$ Decoherence + Black object degeneracy

Entanglement entropy for BPS black objects

Geometric perspective



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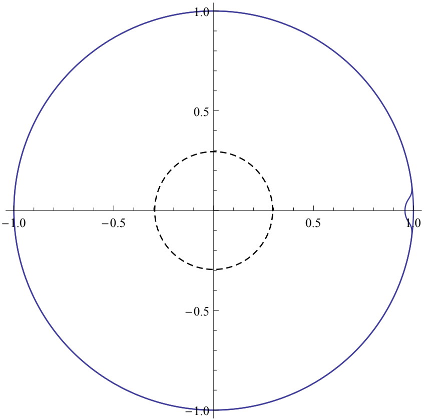

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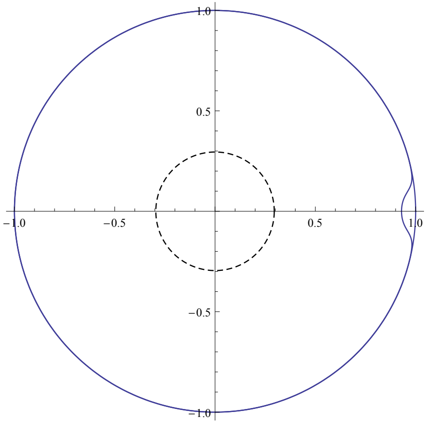

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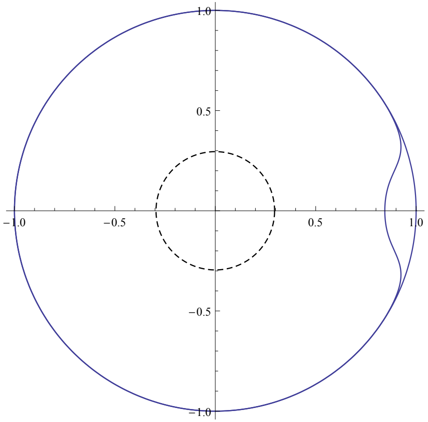

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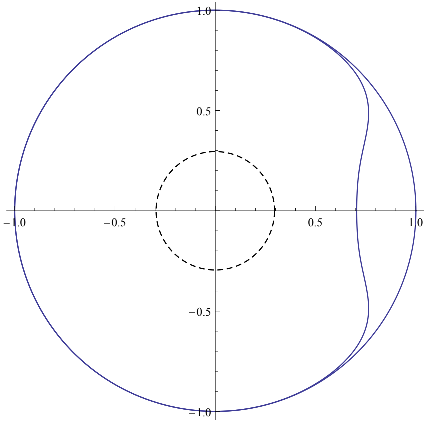

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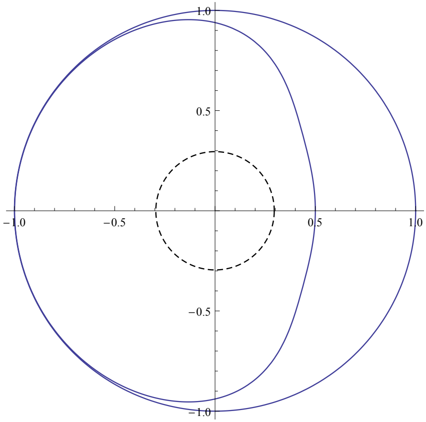

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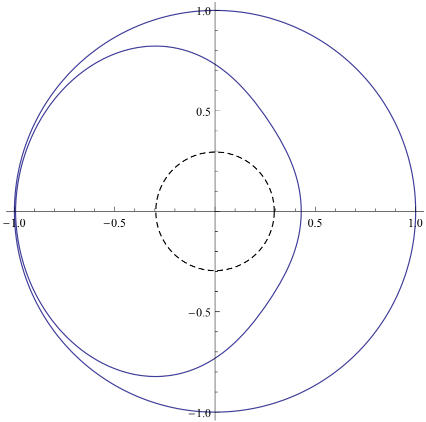
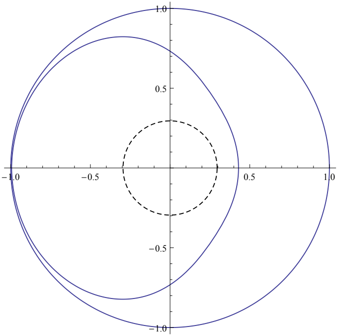


Entanglement entropy for BPS black objects



REE is monotonically decreasing up to a scale set

by symplectic combinations of charge/FI

$\mathcal{S}_{min}$ depends only upon the NH AdS_2 radius

Epilogue

Ongoing projects

REE for microstate geometries

S. Haque, V. Jejjala and AV-O

REE for AFlat black holes

S. M. Hosseini and AV-O

Thank you for your attention!!