Aspects of Renormalization Group flow: the a-theorem for gauge theories

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Outline

What is the a-theorem?

Consider a QFT with couplings g_I at energy scale μ.

- For 2D QFT, Zamolodchikov proved $\exists\, c(\mu, g_I)$ decreasing monotonically under RG flow, where $c(\mu, g_I^*) = c$, the central charge of a CFT at RG fixed point g_I^*.
- Cardy asks if a similar situation holds in 4D, either $\exists\, a(\mu, g_I)$ with monotonic behaviour under RG flow (*Strong* a-theorem) or $\exists\, a(\mu, g_I)$ satisfying $(a_{UV} - a_{IR})|_{g_I^*} > 0$ (*Weak* a-theorem).
- Name a-theorem comes from only possible candidate: $\langle T^\mu_\mu \rangle = -\mathbf{c}R$ in 2D, $\langle T^\mu_\mu \rangle = cF - \frac{1}{4}\mathbf{a}G - \frac{1}{72}gR^2 + \ldots$ in 4D.

Why bother?

- Flowing towards IR, $a(\mu, g_I)$ should decrease and approach a new RG fixed point, defining a low-energy effective theory with less "degrees of freedom".
- Monotonic flow will help address the possibiliy of limit cycles or chaotic behaviour in RG flows, showing they cannot occur for a renormalizable 4D QFT.
- In 2D the c-theorem shows scale-invariance implies conformal invariance; a-theorem may give insight as to whether the same is true for 4D.

What's been done?

- 1990: Jack and Osborn established criteria for the a-theorem to hold.
- 1998: Osborn and Freedman showed the a-theorem holds perturbatively for sufficiently weak coupling.
- 2004: Intriligator *et al* proposed an explicit function a for Supersymmetric theories.
- 2011: Komargodski and Schwimmer proposed a proof for the weak a-theorem based on a 4-dilaton amplitude.
- 2014: Jack and Osborn used the a-theorem as a hypothesis to explicitly calculate 3-loop quantities and provide restrictions on their form for general theories. *This will be the main focus of the talk.*

The theoretical preamble

- Take a theory with couplings $\{g^I\}$. $\exists\, A$ such that $\forall\, \bar{g} \in \{g^I\}$, $\partial_I A = T_{IJ}\beta^J$.
- For RG fixed point g^*, $\beta^I(g^*) = 0$ and $\frac{1}{4}A = a$. For non-RG fixed points, A arbitrary up to $\beta^I G_{IJ}\beta^J$.
- $G_{IJ} = T_{(IJ)} = \frac{1}{2}(T_{IJ} + T_{JI})$ satisfies $\beta^I\partial_I A = \beta^I G_{IJ}\beta^J$, expressing $\mu\frac{d}{d\mu}A = \beta^I\partial_I A$ shows flow is monotonic.
- If $\exists$ some positive-definite G_{IJ}, the strong a-theorem holds.
- Multiplying by dg^I gives $dA = dg^I T_{IJ}\beta^J$, i.e. first order differential equation for each coupling $d_{\bar{g}}A = dg^{\bar{g}}T_{\bar{g}\bar{g}}\beta^{\bar{g}}$: solve for A.
- Note, *DRED* is used instead of *DREG* in order to facilitate comparison with Supersymmetry, but *DREG* is perfectly viable.

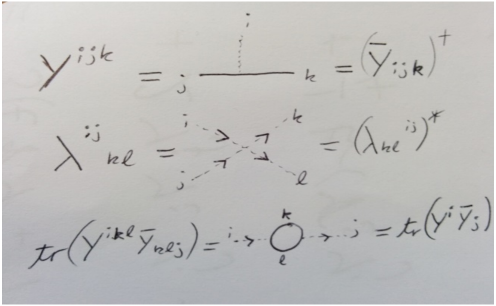
Non-supersymmetric theories
The non-gauge case

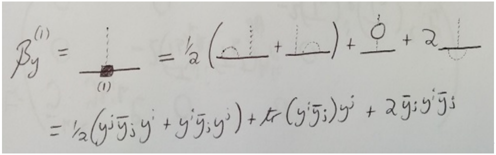
Let $\mathcal{L} = \partial\phi^* \cdot \partial\phi + i\bar{\psi}\sigma \cdot \partial\psi + i\chi\bar{\sigma} \cdot \partial\chi - y\bar{\chi}\phi\psi - \bar{y}\bar{\psi}\bar{\phi}\chi - \frac{\lambda}{4}(\bar{\phi}\phi)^2$

- Couplings $\{g^I\} = \{y, \lambda\}$ $\left(\bar{y} = (y)^\dagger\right)$
- Need Yukawa and Scalar β-functions
- Assume $T_{IJ} = G_{IJ}$ is symmetric (works to all orders so far)
- Lowest order A is $A^{(3)}$: solve $d_y A^{(3)} = dg^y \; G^{(2)}_{y\bar{y}} \; \beta^{\bar{y}\,(1)}$
- Next lowest is $A^{(4)}$: solve
  $d_y A^{(4)} = dg^y \; G^{(2)}_{y\bar{y}} \; \beta^{\bar{y}\,(2)} + dg^y \; G^{(3)}_{y\bar{y}} \; \beta^{\bar{y}\,(1)}$ and
  $d_\lambda A^{(4)} = dg^\lambda \; G^{(3)}_{\lambda\lambda^*} \; \beta^{\lambda^*\,(1)}$
- To avoid endless contraction indices, use diagrammatical notation for couplings.

Non-supersymmetric theories

Diagrammatical notation

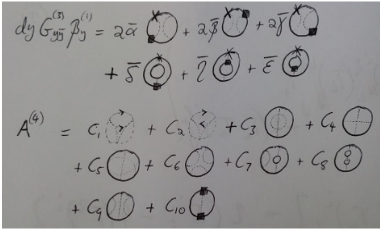




Non-supersymmetric theories

Solutions and constraints

- $G^{(2)}_{y\bar{y}}$ is given, by an explicit curved spacetime calculation, to be $\frac{1}{3}\circ$, where $y \circ \bar{y} \equiv y^{ijk}\bar{y}_{ijk}$. Using an ansatz with arbitrary coefficients, it turns out $A^{(3)} = \frac{1}{12} y \circ \beta^{\bar{y}\,(1)}$.
- Similarly, $G^{(3)}_{\lambda\lambda^*} = \frac{1}{24}\circ$, and so an ansatz for $G^{(3)}_{y\bar{y}}$ and $A^{(4)}$ is:



Non-supersymmetric theories
Solutions and constraints

- Somewhat strangely, there is a solution to $A^{(4)}$ despite $G^{(3)}_{y\bar{y}}$ not being uniquely specified:

$$c_1 = \frac{1}{72} \quad c_2 = \frac{1}{18} \quad c_3 = \frac{1}{24} \quad c_4 = -\frac{1}{6} \quad c_5 = \frac{1}{9}$$
$$c_6 = \frac{1}{72} \quad c_7 = -\frac{1}{6} \quad c_8 = -\frac{1}{36} \quad c_9 = 0 \quad c_{10} = \bar{\alpha}$$

- The coefficients of $G^{(3)}_{y\bar{y}}$ are restricted to obey the following equality:

$$2(\bar{\beta} + \bar{\gamma}) = 4\bar{\alpha} + \frac{1}{6} = 2\alpha + \delta + \frac{1}{2} = \eta + \epsilon$$

- c_{10} is completely arbitrary, and so is chosen to be $\bar{\alpha}$ for convenience.

Non-supersymmetric theories

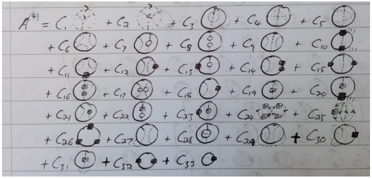
Generalizing to gauge theories

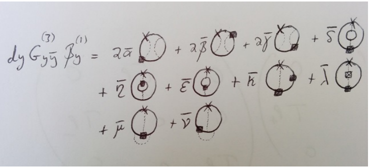
Extend $\mathcal{L}_\partial \rightarrow \mathcal{L}_D - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$ with $\left[T_a, T_b\right] = f_{abc} T_c$.

- A new coupling $g \in \{g^I\}$ appears, so now $\{g^I\} = \{y, \lambda, g\}$
- Add new notation for Casimir elements
- $A^{(3)}$ generalizes easily; no new $G^{(2)}_{y\bar{y}}$ terms necessary
- Use $d_g A^{(3)} = G^{(2)}_{gg}\ \beta^{g\,(1)} + G^{(1)}_{gg}\ \beta^{g\,(2)}$ to find Yukawa terms in $G^{(2)}_{gg}$ and (more importantly) $\beta^{g\,(2)}$, knowing only $\beta^{g\,(1)}$
- Full $\beta^{g\,(2)}$ should follow from adding pure gauge terms to $A^{(3)}$, coefficients possibly fixed by consistency with Supersymmetry
- $A^{(4)}$ is much more difficult...

Non-supersymmetric theories

Generalizing to gauge theories





- $G^{(3)}_{\lambda\lambda^*}$ does not change, but $G^{(3)}_{y\bar{y}}$ needs more gauge-dependent terms
- Exact solution for all $A^{(4)}$ coefficients (except pure gauge terms) has been obtained, consistency conditions for $G^{(3)}_{y\bar{y}}$ extended to:

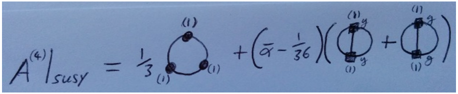
$$2(\bar{\beta}+\bar{\gamma}) = 4\bar{\alpha}+\frac{1}{6} = 2\alpha+\delta+\frac{1}{2} = \eta+\epsilon$$
$$= \frac{1}{3}\bar{\kappa}-\frac{1}{18} = \bar{\mu}-\frac{1}{2} = \frac{1}{2}\bar{\nu}-\frac{1}{6}$$

Supersymmetric theories

Do the calculations match?

Supersymmetric theories work pretty much the same way, using the same structures G_{IJ} adapted to the supersymmetric case. For a general $\mathcal{N} = 1$ Wess-Zumino theory with a chiral (and anti-chiral) superfield,

- $\lambda \equiv \lambda(y, \bar{y}, g)$, so $\{g^I\} = \{y, \bar{y}, g\}$
- Anomalous dimension $\gamma^{(1)} = \frac{1}{2} y^{imn} \bar{y}_{mnj} + 2g^2 (T^2)^i{}_j$
- $\beta_y = (\gamma * y)^{ijk} \equiv (\gamma)^i{}_m y^{mjk} + (\gamma)^j{}_m y^{imk} + (\gamma)^k{}_m y^{ijm}$
- Performing the same procedure gives:



Supersymmetric theories
The Λ-equation

There is a potential, nonperturbative A candidate for SUSY gauge theories with n_c chiral superfields:

- $A = \frac{1}{12}n_c - \frac{1}{2}tr(\gamma)^2 + \frac{1}{3}tr(\gamma)^3 + \Lambda \circ \beta_{\bar{y}} + \beta_y \circ H \circ \beta_{\bar{y}} + \Xi(g)\beta_g$

If this holds, then calculating $\partial_y A$ and requiring $\partial_y A = T_{IJ}\beta^J$ forces (for some calculable constant x)

$$\bar{y} \cdot \Lambda = \gamma - \gamma^2 + \Theta \circ \beta_{\bar{y}} + x\Xi(g)$$

This is the Λ-equation, and allows restrictions on the form of anomalous dimensions for $\mathcal{N} = 1$ SUSY. Specialising to $\mathcal{N} = 2$ and exploiting one-loop finiteness gives many more constraints at higher orders.

Supersymmetric theories

The 3-loop Anomalous Dimension

- Similar to *A*, one can calculate perturbatively
- $3\bar{y} \cdot \Lambda^{(1)} = \gamma^{(1)}$
- $3\bar{y} \cdot \Lambda^{(2)} = \gamma^{(2)} - \gamma^{(1)2} + \Theta^{(1)} \beta_{\bar{y}}^{(1)} + \dots$
- $3\bar{y} \cdot \Lambda^{(3)} = \gamma^{(3)} - \gamma^{(2)}\gamma^{(1)} - \gamma^{(1)}\gamma^{(2)} + \Theta^{(2)} \beta_{\bar{y}}^{(2)} + \dots$

$\gamma^{(3)}$ has transcendental terms with factor $\zeta(3)$, can separate into separate sets of equations with arbitrary coefficients. Solution gives a set of 20 constraints; not quite enough to completely solve for $\gamma^{(3)}$.

Summary

- Complete calculation (up to purely gauge terms) of $A^{(4)}$ for theory with a complex scalar, two chiral fermions and single gauge group. Verified by comparison with both general 3-loop gauge β-function and Standard Model 3-loop gauge β-function.
- Generalized coefficient restrictions in $G^{(3)}_{y\bar{y}}$ to the gauge case.
- Demonstrated restriction on the form of $\gamma^{(3)}$ via the Λ-equation; almost solvable, constraints satisfied by actual general $\gamma^{(3)}$ calculation.

Summary

Further work:

- Extend to several scalars/fermions, massive theories.
- Finish translation of non-supersymmetric $A^{(4)}$ to the supersymmetric case; have verified $\beta^{y} \circ \beta^{\bar{y}}$ works as required.