

The Quantum Closet

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- G. Bertoldi, AEF & M. Matone, CQG 17 (2000) 3925.
- AEF, Marco Matone, arXiv:1211.0798.
- AEF, arXiv:1204.3185 ; 1305.0044.

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- Motivation – quantum gravity

- The Equivalence Postulate \Rightarrow QSHJE

- \rightarrow Schrödinger eq.

- The Equivalence Postulate \Rightarrow CoCycle Condition

- \rightarrow Möbius invariance

- The Equivalence Postulate \Rightarrow Energy quantization

- & Time Parameterisation

- Möbius invariance \Rightarrow Compact universe

- Conclusions

Motivation General Relativity: → Covariance & Equivalence Principle
→ fundamental geometrical principle

Quantum Mechanics: No Such Principle
Axiomatic formulation ... $P \sim |\Psi|^2$

However Quantum + Gravity Theory
not known

Main effort: quantize GR; quantize space–time: *e.g.* superstring theory

The main successes of string theory:

- 1) Viable perturbative approach to quantum gravity
- 2) Unification of gravity, gauge & matter structures
i.e. construction of phenomenologically realistic models
→ relevant for experimental observation

State of the art: MSSM from string theory
(AEF, Nanopoulos, Yuan, NPB 335 (1990) 347)
(Cleaver, AEF, Nanopoulos, PLB 455 (1999) 135)

Adaptation of Hamilton–Jacobi theory

Hamilton's equations of motion $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$

$$H(q, p) \longrightarrow K(Q, P) \equiv 0 \quad \Longrightarrow \quad \dot{Q} = \frac{\partial K}{\partial P} \equiv 0, \quad \dot{P} = -\frac{\partial K}{\partial Q} \equiv 0$$

The solution is the Classical Hamilton–Jacobi Equation

$$H(q, p) \longrightarrow K(Q, P) = H\left(q, p = \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0 \quad \Rightarrow \quad \text{CHJE}$$

stationary case $\longrightarrow \frac{1}{2m} \left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E = 0$

$(q, p) \rightarrow (Q, P)$ via canonical transformations

q, p are independent. Solve. Then $p = \frac{\partial S}{\partial q}$

Quantum mechanics: $[\hat{q}, \hat{p}] = i\hbar \rightarrow q, p \rightarrow$ not independent

Assume $H \rightarrow K$ i.e. $W(Q) = V(Q) - E = 0$ always exists

But q, p not independent. $p = \frac{\partial S}{\partial q}$.

Equivalence postulate:

Consider the transformations on

$$\left(q, S_0(q), p = \frac{\partial S_0}{\partial q} \right) \longrightarrow \left(\tilde{q}, \tilde{S}_0(\tilde{q}), \tilde{p} = \frac{\partial \tilde{S}_0}{\partial \tilde{q}} \right)$$

Such that

$$W(q) \longrightarrow \tilde{W}(\tilde{q}) = 0$$

exist for all $W(q)$

\implies QHJE

\longrightarrow Schrödinger equation

Implies: Covariance of HJE

But:
$$\frac{1}{2m} \left(\frac{\partial S_0}{\partial q} \right)^2 + V(q) - E = 0$$

Is not covariant under $q \rightarrow \tilde{q}(q)$.

Further: $W(q) \equiv 0$ is a fixed state under $q \rightarrow \tilde{q}(q)$.

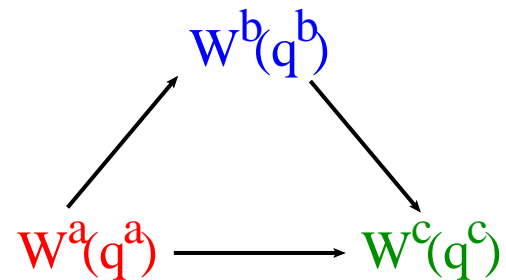
Assume:
$$\frac{1}{2m} \left(\frac{\partial S_0}{\partial q} \right)^2 + W(q) + Q(q) = 0$$

The most general transformations

$$\tilde{W}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q} \right)^{-2} W(q) + (\tilde{q}; q),$$
$$\tilde{Q}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q} \right)^{-2} Q(q) - (\tilde{q}; q),$$

with $\tilde{S}_0(\tilde{q}) = S_0(q)$ under $q \rightarrow \tilde{q} = \tilde{q}(q)$

With: $W^0(q^0) = 0 \rightarrow$ All: $W(q) = (q; q^0)$



Cocycle Condition: $(q^a; q^c) = \left(\frac{\partial q^b}{\partial q^c} \right)^2 \left[(q^a; q^b) - (q^c; q^b) \right]$

\Rightarrow Theorem $(q^a; q^c)$ invariant under Möbius transformations $\gamma(q^a)$

In 1D: $(q^a; q^c) \sim \{q^a; q^c\}$ Uniquely

Schwarzian derivative $\{h(x); x(y)\} = \left(\frac{\partial y}{\partial x} \right)^2 \{h(x); y\} - \left(\frac{\partial y}{\partial x} \right)^2 \{x; y\}.$

$$U(q) = \{h(q); q\} = \left\{ \frac{Ah + B}{Ch + D}; q \right\}$$

Invariant under Möbius transformations

Identity

$$\left(\frac{\partial S_0}{\partial q}\right)^2 = \frac{\beta^2}{2} \left(\left\{ e^{\frac{i2S_0}{\beta}}; q \right\} - \{S_0; q\} \right)$$

Make the following identifications

$$W(q) = -\frac{\beta^2}{4m} \left\{ e^{\frac{i2S_0}{\beta}}; q \right\} = V(q) - E$$

$$Q(q) = \frac{\beta^2}{4m} \{S_0; q\}$$

The Modified Hamilton–Jacobi Equation becomes

$$\frac{1}{2m} \left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m} \{S_0; q\} = 0$$

in the limit $\beta \rightarrow 0$ we get back the CSHJE and $S_0^{cl} = \lim_{\beta \rightarrow 0} S_0$

From the properties of the SD $\{; \}$

$$V(q) - E = -\frac{\beta^2}{4m} \left\{ e^{\frac{i2S_0}{\beta}}; q \right\}$$

is a potential of the 2^{nd} -order diff. Eq.

$$\left(-\frac{\beta^2}{2m} \frac{\partial^2}{\partial q^2} + V(q) - E \right) \Psi(q) = 0 \quad \Rightarrow \quad \beta = \hbar$$

The general solution

$$\Psi(q) = \frac{1}{\sqrt{S'_0}} \left(A e^{+\frac{i}{\hbar} S_0} + B e^{-\frac{i}{\hbar} S_0} \right)$$

and

$$e^{+\frac{i2S_0}{\hbar}} = e^{i\alpha} \frac{w + i\bar{l}}{w - il} \quad w = \frac{\psi_1}{\psi_2}$$

$$l = l_1 + i l_2 \quad l_1 \neq 0 \quad \alpha \in R$$

The equivalence transformation

$$W(q) = V(q) - E \longrightarrow \tilde{W}(\tilde{q}) = 0$$

always exists

We have to find $q \rightarrow \tilde{q}$ take $\tilde{q} = \frac{\psi_1}{\psi_2}$

$$\text{then } \left(-\frac{\beta^2}{2m} \frac{\partial^2}{\partial q^2} + V(q) - E \right) \Psi(q) = 0 \longrightarrow \frac{\partial^2}{\partial \tilde{q}^2} \tilde{\psi}(\tilde{q}) = 0$$

$$\text{where } \tilde{\psi}(\tilde{q}) = \left(\frac{dq}{d\tilde{q}} \right)^{-\frac{1}{2}} \psi(q)$$

Energy quantization:

Probability: $\implies (\Psi, \Psi')$ continuous ; $\Psi \in L^2(\mathbb{R})$

\implies quantization, bound states

What are the conditions on the trivializing transformations?

$$q^0 = w = \frac{\psi_1}{\psi_2} = \frac{\psi^D}{\psi}$$

we have $\{w, q\} = -\frac{4m}{\hbar^2}(V(q) - E)$

$\implies w \neq \text{const}$; $w \in C^2(\mathbb{R})$ and w'' differentiable on \mathbb{R}

In addition from the properties of $\{, \}$ $\rightarrow \{w, q^{-1}\} = q^4\{w, q\}$

$\implies w \neq \text{const}$; $w \in C^2(\hat{R})$ and w'' differentiable on \hat{R}

where $\hat{R} = \mathbb{R} \cup \{\infty\}$



Equivalence postulate \implies continuity of (ψ^D, ψ) and $(\psi^{D'}, \psi')$

Theorem:

$$\text{if } V(q) - E = \begin{cases} P_-^2 > 0 & \text{for } q < q_- \\ P_+^2 > 0 & \text{for } q > q_+ \end{cases}$$

then the ratio $w = \psi^D / \psi$ is continuous on \hat{R}

iff the Schrödinger equation admits an $L^2(\mathbb{R})$ solution

$$1) \quad \psi \in L^2(\mathbb{R}) \implies \psi^D \notin L^2(\mathbb{R})$$

$$w = \frac{A\psi_D + B\psi}{C\psi_D + D\psi} \implies \lim_{q \rightarrow \pm\infty} w = \frac{A}{C}$$

$$\implies w(-\infty) = w(+\infty)$$

2) ...

Potential Well:

$$V(q) = \begin{cases} 0 & |q| \leq L \\ V_0 & |q| > L \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \quad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$\begin{array}{lll} |q| \leq L & \Psi_1^1 = \cos kq & \Psi_2^1 = \sin kq \\ q > L & \Psi_1^2 = e^{-Kq} & \Psi_2^2 = e^{Kq} \end{array}$$

take (1, 1) : Ψ, Ψ' continuous $\Rightarrow k \tan kL = K$

$$\Rightarrow w = \frac{1}{[k \sin(2kL)]} \begin{cases} \cos(2kL) - e^{-2K(q+L)} & q < -L \\ \sin(2kL) \tan(kq) & |q| \leq L \\ e^{2K(q-L)} - \cos(2kL) & q > L \end{cases}$$

$$\lim_{q \rightarrow \pm} \frac{\psi^D}{\psi} = \pm\infty \quad \Longrightarrow \quad E_n(k \tan kL = K) \text{ are admissible solutions}$$

$$\text{take (1, 2) : } \Psi, \Psi' \text{ continuous} \quad \Longrightarrow \quad k \tan(kl) = -K$$

$$\Longrightarrow w = \frac{1}{[k \sin(2kL)]} \begin{cases} \cos(2kL) - e^{2K(q+L)} & q < -L \\ \sin(2kL) \tan(kq) & |q| \leq L \\ e^{-2K(q-L)} - \cos(2kL) & q > L \end{cases}$$

$$\lim_{q \rightarrow \pm} \frac{\psi^D}{\psi} = \mp \frac{1}{k} \cot(2kL) \quad \Longrightarrow \quad w(-\infty) \neq w(+\infty)$$

$(k^{-1}(\cot 2kL) = 0$ is not compatible with $k \tan(kL) = -K$)

$$\Longrightarrow E_n(k \tan kL = -K) \text{ are not admissible solutions}$$

Time parameterisation

Bohmian mechanics : $p = \frac{\partial S}{\partial q} = m\dot{q} \Rightarrow$ Trajectory representation

Jacobi time : $t = \frac{\partial S_0}{\partial E}$.

In classical mechanics: Jacobi time = Mechanical time

$$t - t_0 = m \int_{q_0}^q \frac{dx}{\partial_x S_0^{\text{cl}}} = \int_{q_0}^q dx \frac{\partial}{\partial E} \partial_x S_0^{\text{cl}} = \frac{\partial S_0^{\text{cl}}}{\partial E}.$$

In Quantum HJ Theory: Jacobi time \neq Mechanical time

$$t - t_0 = \frac{\partial \mathcal{S}_0^{\text{qm}}}{\partial E} = \frac{\partial}{\partial E} \int_{q_0}^q dx \partial_x \mathcal{S}_0^{\text{qm}} = \left(\frac{m}{2}\right) \int_{q_0}^q dx \frac{1 - \partial_E Q}{(E - V - Q)^{1/2}}$$

$$\Rightarrow m \frac{dq}{dt} = m \left(\frac{dt}{dq}\right)^{-1} = \frac{\partial_q \mathcal{S}_0^{\text{qm}}}{(1 - \partial_E \mathcal{V})} \neq \frac{\partial \mathcal{S}^{\text{qm}}}{\partial q}$$

Floyd: Use Jacobi theorem to define time \rightarrow trajectories

Energy quantisation \Leftrightarrow Compact space

\Rightarrow Floyd time is ill defined for the QHJT

No Trajectories in EPoQM

Quantum potential as a curvature term:

Using the property of the Schwarzian derivative

$$\{S_0; q\} = - \left(\frac{\partial S_0}{\partial q} \right)^2 \{q; S_0\},$$

We can rewrite the Quantum Stationary Hamilton Jacobi Equation as

$$\frac{1}{2m} \left(\frac{\partial S_0}{\partial \hat{q}} \right)^2 + V(\hat{q}) - E = 0,$$

where

$$\hat{q} = \int^q \frac{dx}{\sqrt{1 - \frac{\hbar^2}{2} \{q; S_0\}}}.$$

Flanders: J. Diff. Geom. 1970, 575 $\rightarrow \{ ; \}$ \rightarrow a curvature term

In higher dimensions $Q(q) \sim \frac{\Delta R(q)}{R} \rightarrow$ curvature of $R(q)$

Length Scale

For $W^0(q^0) = 0$

$$\frac{\partial^2 \Psi}{\partial q^2} = 0 \Rightarrow \psi_1 = q^0 \quad ; \quad \psi_2 = \text{const}$$

\Rightarrow duality implies a length scale

$$\Rightarrow e^{\frac{2i}{\hbar} S_0^0} = e^{i\alpha \frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0}},$$

$$p_0 = \frac{\partial S_0^0}{\partial q^0} = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}.$$

$$\text{Max}|p_0| = \frac{\hbar}{\text{Re}\ell_0} \rightarrow \text{Re}\ell_0 \neq 0 \rightarrow \text{ultraviolet cutoff}$$

$$\lim_{\hbar \rightarrow 0} p_0 = 0 \Rightarrow \text{Re}\ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}}.$$

$$\ell_0 = \lambda_p \longrightarrow \text{choice consistent with the classical limit}$$

$$Q^0 = \frac{\hbar^2}{4m} \{S_0^0, q^0\} = -\frac{\hbar^2 (\text{Re } \ell_0)^2}{2m} \frac{1}{|q^0 - i\ell_0|^4}.$$

Consistency $\implies q^0 = \psi^D / \psi$ is continuous on $\hat{R} = R \cup \{\infty\}$

Taking $m \sim 100 \text{ GeV};$

$$\text{Re } \ell_0 = \lambda_p \approx 10^{-35} m;$$

$$q^0 \sim 93 \text{ Ly},$$

$$\implies |Q| \sim 10^{-202} \text{ eV}.$$

Generalizations:

Under $(q, S_0(q), p = \frac{\partial S_0}{\partial q}) \longrightarrow (q^v, S_0^v(q^v), p^v = \frac{\partial S_0^v}{\partial q^v})$,

$$p^v = \frac{\partial S^v(q^v)}{\partial q_j^v} = \frac{\partial S(q)}{\partial q_j^v} = \sum_i \frac{\partial S(q)}{\partial q_i} \frac{\partial q_i}{\partial q_j^v}, = J^v p, \quad \text{where} \quad J_{ij}^v = \frac{\partial q_i}{\partial q_j^v}$$

$$\text{with} \quad (p^v|p) \equiv \frac{|p^v|^2}{|p|^2} = \frac{p^{vT} p^v}{p^T p} = \frac{p^T J^{vT} J^v p}{p^T p}.$$

Cocycle condition $\rightarrow (q^a; q^c) = (p^c|p^b) \left[(q^a; q^b) - (q^c; q^b) \right]$.

invariant under D-dimensional Möbius (conformal) trans.

Quadratic identity:

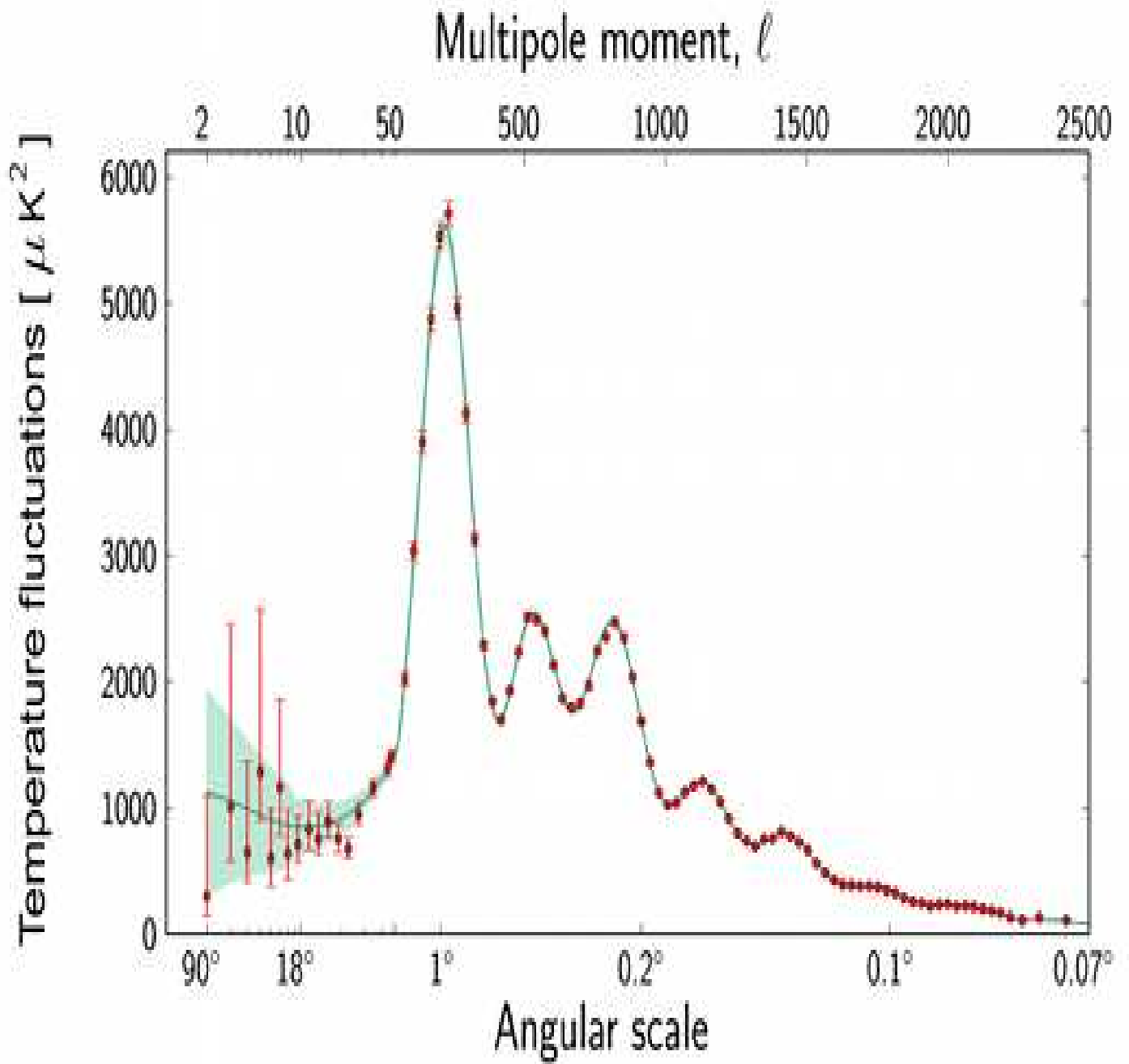
$$\alpha^2(\nabla S_0) \cdot (\nabla S_0) = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \alpha \left(2 \frac{\nabla R \cdot \nabla S_0}{R} + \Delta S_0 \right),$$

or

$$\alpha^2(\partial S) \cdot (\partial S) = \frac{\partial^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \alpha \left(2 \frac{\partial R \cdot \partial S}{R} + \partial^2 S \right),$$

or

$$\alpha^2(\partial S - eA) \cdot (\partial S - eA) = \frac{D^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \frac{\alpha}{R^2} \partial \cdot \left(R^2(\partial S - eA) \right),$$
$$D^\mu = \partial^\mu - \alpha e A^\mu$$



conclusions :

The equivalence postulate $\implies S_0 \neq \text{const} \Leftrightarrow \hbar \neq 0$

$$\Psi(q) = \frac{1}{\sqrt{S'_0}} \left(A e^{+\frac{i}{\hbar} S_0} + B e^{-\frac{i}{\hbar} S_0} \right)$$

$\text{Re} \ell_0 = \lambda_P \rightarrow$ fundamental length scale

$Q(q)$ Intrinsic curvature terms of elementary particles $\neq 0$ Always

CoCyCle Condition: Invariant under Möbius transformations in

$$\hat{R}^D = R^D \cup \{\infty\} \rightarrow \text{Compact Space}$$

Decompactification limit $\Leftrightarrow Q(q) \rightarrow 0 \Leftrightarrow$ classical limit

