Structure Function Resummation and All-α_s-Order Results

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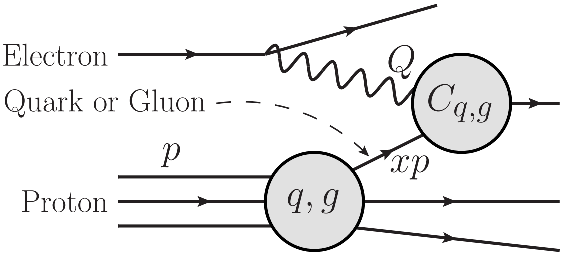
INTRODUCTION

Deep Inelastic Scattering – electron scatters from a proton.



DIS Cross-section: *Structure Functions* F_i,

$$F_i = C_{i,q} \otimes q + C_{i,g} \otimes g$$

C_i, *Coefficient Functions* (perturbative)
q, g, *Parton Distribution Functions* (PDFs) (non-perturbative)

Measure F_i, extract q, g – essential for hadron physics (LHC).

PHYSICAL KERNELS

Splitting Functions (P) (perturbative) describe the energy-scale evolution of the PDFs ($q, g \to f_a$)

$$\frac{df_a}{d\ln Q^2} = P_{ab} \otimes f_b.$$

Physical Kernels (K) describe the energy-scale dependence of F_i,

$$\frac{dF_i}{d\ln Q^2} = K_{ij} \otimes F_j = \left(\beta_{QCD}(\alpha_s)\frac{dC_{i,k}}{d\alpha_s} + C_{i,l}P_{lk}\right)\left(C^{-1}\right)_{kj} \otimes F_j.$$

F_i depend on both PDFs (“flavour singlet”) – evolution is coupled – K is therefore a matrix.

Can compute K contributions up to α_s^3 using known C, P.
[Zijlstra, van Neerven: 1991]
[Moch, Vermaseren, Vogt: 2004]

Large-x Behaviour

In the large-x limit ($x \to 1$), have threshold logarithms $L = \ln(1-x)$.

C, P are *Double Log Enhanced*. Have terms like $\alpha_s^n L^{2n}$.

K *Single Log Enhanced* ($\alpha_s^n L^n$). Conjecturing that this holds to all orders, can predict α_s^4 contributions to C and P.

For example, for the (F_2, F_ϕ) system

$$\begin{aligned} K_{2\phi} = &+ \alpha_s\,(1) \\ &+ \alpha_s^2\left(1 + L + \cancel{L^2}\right) \\ &+ \alpha_s^3\left(1 + L + L^2 + \cancel{L^3} + \cancel{L^4}\right) \\ &+ \alpha_s^4\left(1 + L + L^2 + L^3 + L^4 + L^5 + \cancel{L^6} + P_{qg}^{(3)}\right). \end{aligned}$$

Can "solve" for Leading Logs of $P_{qg}^{(3)}$ (NLL and NNLL).

RESULTS

[Soar, Vogt, Moch, Vermaseren: 2009]

$$P_{qg}^{(3)} \sim +L^6 \Big[0 \Big]_{LL}$$
$$+L^5 \Big[+C_{AF}^3 n_f \left(\frac{32}{27x} + \frac{46}{27} + \frac{52}{27}x - 4x^2 + \frac{16}{9}(1+4x)\ln(x) \right)$$
$$+ C_{AF}^2 C_F n_f \left(\frac{32}{27x} + \frac{58}{27} + \frac{100}{27}x - \frac{68}{9}x^2 + \frac{8}{9}(1+10x)\ln(x) \right)$$
$$+C_{AF}^2 n_f^2 \left(-\frac{4}{27}\left(1-2x+2x^2\right) \right) \Big]_{NLL} + L^4 \Big[\cdots \Big]_{NNLL} .$$

where $C_{AF} = (C_A - C_F)$.

Leading logarithms of α_s^4 contributions to many other quantities determined this way.

UNFACTORIZED STRUCTURE FUNCTIONS

$$F = C \otimes f$$

has been *mass-factorized*. Divergences arising from initial state collinear radiation have been absorbed into the (non-pert.) PDF.

Dimensional Regularization yields, in $D = 4 - 2\epsilon$ dimensions,

$$F = T(\epsilon) \otimes \hat{f}.$$

T contains poles in the ϵ parameter. We can write in the form

$$F = CZ \otimes \hat{f}$$

defining the *Coefficient Function* $C = 1 + \epsilon + \epsilon^2 + \cdots$ and the *Renormalization Function* $Z = \frac{1}{\epsilon} + \frac{1}{\epsilon^2} + \cdots$.

Then, redefine PDF: $f = Z\hat{f}$. Poles have disappeared!

SOLVING FOR Z

Z has energy scale dependence, again governed by the Splitting Functions $P = \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \cdots$.

We have that

$$\frac{dZ}{d\ln Q^2} = \frac{dZ}{d\alpha_s}\left(-\epsilon\alpha_s - \beta_{QCD}\left(\alpha_s\right)\right) = PZ.$$

Solving order-by-order (in α_s expansion) for Z,

$$Z = 1 + \alpha_s \frac{1}{\epsilon} P^{(0)} + \alpha_s^2 \left(\frac{1}{2\epsilon^2}\left(P^{(0)} - \beta_0\right) P^{(0)} + \frac{1}{2\epsilon} P^{(1)}\right) + \mathcal{O}\left(\alpha_s^3\right).$$

Note: The *highest* order poles come with the *lowest* order contributions to P.

These are known. We therefore know the highest poles of Z, and therefore T, to *all* orders in α_s.

Known Terms of T

Double expansion of T in (α_s, ϵ),

$T^{(n,i)}$	ϵ^{-7}	ϵ^{-6}	ϵ^{-5}	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ^{1}	ϵ^{2}
α_s^1	0	0	0	0	0	0	✓	✓	✓	×
α_s^2	0	0	0	0	0	✓	✓	✓	×	×
α_s^3	0	0	0	0	✓	✓	✓	×	×	×
α_s^4	0	0	0	✓	✓	✓	×	×	×	×
	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$

E.g.: C, P known to 3 orders $\rightarrow$ lowest 3 ϵ powers known (✓)

$\times$ terms depend on higher-order contributions to C, P.

SO FAR...

Unfactorized Structure Function: $T = CZ$.

Coefficient Function C: only +ve powers of ϵ. Terms $\alpha_s^i \epsilon^j C^{(i,j)}$.

Renormalization Function Z: contains poles in ϵ. Have an expression in terms of P. Terms $\alpha_s^i P^{(i-1)}$.

Therefore, T in terms of C, P

$$
\begin{aligned}
T = +\alpha_s^1 &\left(\frac{1}{\epsilon} P^{(0)} + \epsilon^0 C^{(1,0)} + \epsilon^1 C^{(1,1)} + \cdots \right) \\
&+\alpha_s^2 \left(\frac{1}{2\epsilon^2} \left[(P^{(0)} - \beta_0) P^{(0)} \right] + \frac{1}{2\epsilon} \left[P^{(1)} + 2P^{(0)} C^{(1,0)} \right] \right. \\
&\qquad \left. +\epsilon^0 \left[P^{(0)} C^{(1,1)} + C^{(2,0)} \right] + \cdots \right) + \mathcal{O}\left(\alpha_s^3\right)
\end{aligned}
$$

RESUMMATION OF T

All-ϵ-expression not possible in general.

However – in the large-N (large-x) limit it *is*, for double logs.

In x-space at α_s^i: terms like $(1-x)^{-j\epsilon}$, with $j = 1, ..., i$.

In N-space,

$$T\Big|_{\alpha_s^i} = \frac{1}{N\epsilon^{2i-1}} \sum_{j=1}^{i} \left[\left(A^{(i,j)} + \epsilon B^{(i,j)} + \epsilon^2 D^{(i,j)} \right) \exp\left(j\epsilon \ln N \right) \right].$$

α_s^i-th contribution has i free coefficients.

*A*s, *B*s, *D*s will determine LL, NLL, NNLL respectively.

[Almasy, Soar, Vogt: 2011]

Resummation of T

Expanding the exponential yields terms (in blue)

$T^{(n,i)}$	ϵ^{-7}	ϵ^{-6}	ϵ^{-5}	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ^{1}	ϵ^{2}
α_s^1	0	0	0	0	0	0	✓	✓	✓	×
α_s^2	0	0	0	0	0	✓	✓	✓	×	×
α_s^3	0	0	0	0	✓	✓	✓	×	×	×
α_s^4	0	0	0	✓	✓	✓	×	×	×	×
	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$

One additional unknown coefficient at each α_s order, but *two* additional constraints – always over-constrained!

Can determine As, Bs, Ds to all α_s orders, and therefore the resummation gives T to all orders in both α_s and ϵ.

Extracting C and γ

$$T = +\alpha_s^1 \left(\frac{1}{\epsilon} P^{(0)} + \epsilon^0 C^{(1,0)} + \epsilon^1 C^{(1,1)} + \cdots \right)$$
$$+\alpha_s^2 \left(\frac{1}{2\epsilon^2} \left[(P^{(0)} - \beta_0) P^{(0)} \right] + \frac{1}{2\epsilon} \left[P^{(1)} + 2P^{(0)} C^{(1,0)} \right] \right.$$
$$\left. +\epsilon^0 \left[P^{(0)} C^{(1,1)} + C^{(2,0)} \right] + \cdots \right) + \mathcal{O}\left(\alpha_s^3\right)$$

Now we can extract the contributions to the expansions of C and P to “all” orders in α_s, ϵ.

(Actually, we know the contributions to as many orders as we care to compute.)

Can we find a closed formula which reproduces them?

BERNOULLI FUNCTIONS

Useful for writing all-α_s closed expressions. Defined by

$$\mathcal{B}_0(x) = 1 - \frac{x}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{[(2n)!]^2} |B_{2n}| x^{2n},$$

$$\mathcal{B}_k(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!} x^n, \qquad \mathcal{B}_{-k}(x) = \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!} x^n,$$

coefficients B_n are the *first Bernoulli Numbers*, which read (OEIS A027641/A027642)

$$B_0 = 1,\ \ B_1 = -\frac{1}{2},\ \ B_2 = \frac{1}{6},\ \ B_4 = -\frac{1}{30},\ \ B_6 = \frac{1}{42}, \ldots$$

(odd-n $B_n = 0$, for $n \geq 3$).

[Almasy, Soar, Vogt: 2011]

A CLOSED EXPRESSION

Choose a set of Bernoulli Functions. Choose coefficients to reproduce the function we want.

For example, let's try P_{qg} (from resummation of $T_{2,g}$)

$$P_{qg} = \frac{\alpha_s}{N}\left(A_1 + A_2\frac{\mathcal{B}_{-2}}{\tilde{\alpha}_s} + A_3\frac{\mathcal{B}_{-1}}{\tilde{\alpha}_s} + A_4\mathcal{B}_0 + A_5\mathcal{B}_1\right)$$

(Bernoulli Functions have argument $\tilde{\alpha}_s = 4\alpha_s(C_A - C_F)\ln^2 N$).

$$P_{qg} = \left[2\alpha_s n_f \mathcal{B}_0 \frac{1}{N}\right]_{LL} + \left[\alpha_s^2 n_f \frac{\ln N}{N}\left((12C_F - 2\beta_0)\frac{\mathcal{B}_{-1}}{\tilde{\alpha}_s} + \beta_0\frac{\mathcal{B}_{-2}}{\tilde{\alpha}_s} + (6C_F - \beta_0)\mathcal{B}_1\right)\right]_{NLL}$$

With NNLL also, agree with the expression for $P_{qg}^{(3)}$ on Slide 4!

My Results

P_{qg} resummation completed by others.

I have been looking at coefficient and splitting functions for *Lepton-Photon DIS*. New Results:

$$P_{q\gamma} = \left[\frac{-4\mathcal{B}_0\left(\hat{\alpha}_s\right)}{N}\right]_{LL} + \left[-2\alpha_s\frac{\ln N}{N}\left(\frac{\beta_0}{\hat{\alpha}_s}\mathcal{B}_{-2}\left(\hat{\alpha}_s\right) + 2\left(6C_F - \beta_0\right)\frac{1}{\hat{\alpha}_s}\mathcal{B}_{-1}\left(\hat{\alpha}_s\right) + \left(6C_F - \beta_0\right)\mathcal{B}_1\left(\hat{\alpha}_s\right)\right)\right]_{NLL} + \left[\cdots\right]_{NNLL}$$

Also: $C_{2,\gamma}$ to NNLL.

In progress: $P_{g,\gamma}$ and $C_{\phi,\gamma}$.

CONCLUSIONS

Working in D dimensions, we can produce all-α_s-order expressions for large-N leading logarithms of coefficient and splitting functions.

Can use these expressions to verify the conjecture of Single Log Enhancement of Physical Kernels to all α_s orders.

Future work: *Polarised* DIS splitting functions recently completed at NNLO. Extend these methods to the polarised case.
[Moch, Vermaseren, Vogt: Loops and Legs 2014].