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Towards the classification of free-fermionic models in 4D-heterotic string models

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Abstract

We study the classification of free-fermionic heterotic string models. We start by an introduction to string theory, from the bosonic string to the heterotic string, before explaining the free-fermionic formalism used to construct our models. We then determine the tools for a classification of free-fermionic heterotic string vacua in which the $SO(10)$ GUT symmetry is broken at the string level to flipped $SU(5) \times U(1)$ subgroup. We derive the conditions for possible enhancements of this gauge group and determine algebraic expressions for Generalised GSO projections for all sectors, which will allow a computer analysis of the entire spectrum of these models. Finally we analyse the phenomenology of an exophobic $SU(6) \times SU(2)$ heterotic string model that is free of massless exotic fractionally charged states, obtained by enhancement of a Pati-Salam gauge group model. We derive the superpotential and show the existence of F- and D-flat directions that leave Higgs doublets light while giving masses to colour triplets beyond those of the Standard Model.

Résumé

Nous étudions la classification de modèles de corde hétérotique de type fermions libres (*free-fermionic*). Nous commençons par une introduction à la théorie des cordes, depuis la corde bosonique jusqu'à la corde hétérotique, avant de détailler le formalisme *free-fermionic* que nous avons utilisé pour la construction de nos modèles. Nous déterminons ensuite les outils nécessaires à une classification des états du vide de la corde hétérotique pour lesquels le groupe de grande unification $SO(10)$ est brisé à l'échelle des cordes en le sous-groupe $SU(5) \times U(1)$. Nous dérivons les conditions nécessaires à l'obtention d'un élargissement du groupe de jauge et déterminons pour tous les secteurs les expressions des projections GSO généralisées, ce qui permettra une analyse informatique complète du spectre de ces modèles. Pour finir nous analysons la phénoménologie d'un modèle exophobique de groupe de jauge $SU(6) \times SU(2)$ qui ne contient pas d'états exotiques ayant une charge fractionnelle. Ce dernier est obtenu par un élargissement d'un modèle ayant pour groupe de jauge le groupe de Pati-Salam. Nous en dérivons le superpotentiel et démontrons l'existence de directions F-flat et D-flat qui laissent les doublets de Higgs sans masse tout en rendant massifs les triplets de couleur non présents dans le Modèle Standard.

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Introduction

The Standard Model is often hailed as one of the crowning achievements of twentieth century theoretical physics. For the past 25 years, its predictions have matched experimental data with amazing precision; it has been tested with everything from desktop experiments to huge particle accelerators, and it has yet to be faulted. But we believe it cannot be the end of the story.

Indeed it seems too arbitrary, with so many free parameters, to constitute an accomplished theory of nature. There is some evidence of a possible embedding of the Standard Model in larger grand unified groups, such as SO(10), reducing the number of gauge charge parameters to one (the number of representations needed to accommodate the three generations of the Standard Model). But there remain questions to be answered, such as why three generations or how do we determine the various mass and flavour mixing parameters of the Standard Model? We can seek answers to these questions at an energy scale above the GUT scale. But at such energies gravitational effects are no more negligible compared to other forces, so that we need a theory that encompasses gravity.

String theory provides a self consistent framework for the synthesis of quantum mechanics and gravity. It no longer considers punctual particles but one-dimensional objects, strings, whose oscillation modes correspond to particles. We are here interested in a phenomenological approach that consists of building string models in four dimensions that aim to reproduce the Standard Model.

There are five different 10-dimensional string theories, which are now believed to be limits of a more fundamental theory. Any of these limits can be used to construct string models. We do not expect any of these limits to provide a complete description of the true vacuum, but we can use them to probe some of its properties. The heterotic string theory facilitates the embedding of the Standard Model spectrum in SO(10) and will be the framework of our study. In our phenomenological approach the heterotic string has to be compactified from 10 to 4 dimensions. A class of compactifications that preserve the embedding in SO(10) is based on the $Z_2 \times Z_2$ orbifolds and has been studied using a free-fermionic formulation since the late eighties.

Models in which the $SO(10)$ GUT symmetry is broken at the string level to different subgroups have been suggested and studied using the rules defined by Antoniadis, Bacchas and Kounnas in [1, 2], in particular with a flipped $SU(5) \times U(1)$ gauge group ([5, 4, 3]).

However these attempts remained limited to the study of a few isolated models. Over the past few years tools for the systematic classification of free-fermionic $Z_2 \times Z_2$ orbifolds were developed and in this report we deal with an extension of this classification.

In the first part we will start by an introduction to string theory that will set the framework of the heterotic string. We then explain the free-fermionic formalism and expose the tools used to study string vacua. In the third part we present our work on the classification of free-fermionic models with a flipped $SU(5) \times U(1)$ gauge group, leading to a set of algebraic expressions that enables the analysis of the spectrum of this class of models. Finally the fourth part introduces a new string model with an $SU(6) \times SU(2)$ gauge group and some of its phenomenological properties.

Part 1

From the bosonic string to the heterotic string

In this part we will start by introducing string theory, mainly based on references [8, 17]. We first describe both the bosonic string and the superstring including their quantization. Then we present the heterotic superstring that will be the framework of our work.

1.1 The bosonic string

1.1.1 The relativistic string

The string action

The string propagates in a D-dimensional Lorentzian space-time which sweeps out a two-dimensional surface called the world-sheet. The points on the world-sheet are parametrized by the coordinates $\sigma_0 = \tau \in [-\infty, +\infty]$, which is time-like and $\sigma_1 = \sigma \in [0, \pi]$, which is space-like. The mapping functions $X^\mu(\sigma_0, \sigma_1)$ describe the position of the string in space-time.

The propagation of the string can be described by the the Nambu-Goto action

$$S_{NG} = -T \int d\sigma d\tau \sqrt{-\det G_{\alpha\beta}} \, , \tag{1.1}$$

where $G_{\alpha\beta} = G_{\mu\nu}(X) \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X_\mu}{\partial \sigma_\beta}$ is the induced metric on the world-sheet and T is called the string tension.

In the case of a flat Minkowski space-time, this action can be rewritten as

$$S_{NG} = -T \int d\sigma d\tau \sqrt{\left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau}\right)^2 \left(\frac{\partial X_\mu}{\partial \sigma}\right)^2} \, . \tag{1.2}$$

The Nambu-Goto action describes the area of the world-sheet, so that the motion of the string minimizes the action.

We can also write down an equivalent action, the Polyakov action, by introducing an additional world-sheet metric $h_{\alpha\beta}$

$$S_P = -\frac{T}{2}\int d\sigma d\tau \sqrt{-h} h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X_\mu}{\partial\sigma^\beta} \ , \tag{1.3}$$

where $h = \det h_{\alpha\beta}$. Due to the presence of the square-root in the Nambu-Goto action, the Polyakov action is more convenient for the quantization of the string. From now on, we will thus use the latter.

The Polyakov action can be written in a more convenient way. Using first the reparametrization invariance, the two-dimensional world-sheet metric becomes conformally flat (providing that there is no topological obstruction, which is the case here) :

$$h_{\alpha\beta} = e^{\Lambda(\sigma)}\eta_{\alpha\beta} \ , \tag{1.4}$$

where $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, using the conformal invariance, we obtain

$$h_{\alpha\beta} = \eta_{\alpha\beta} \ . \tag{1.5}$$

This is called the conformal gauge. The Polyakov action becomes

$$S_P = \frac{T}{2}\int d\sigma d\tau \left(\dot{X}^2 - X'^2\right) \ , \tag{1.6}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial\tau}$ and $X'^\mu = \frac{\partial X^\mu}{\partial\sigma}$.

The equation of motion

The equations of motion for the bosonic string are obtained by the variation of the Polyakov action :

$$\delta S_P = T\int d\sigma d\tau \left(\ddot{X} - X''\right)\delta X + T\int d\sigma \left[\dot{X}\delta X\right]_{-\infty}^{+\infty} - T\int d\tau \left[X'\delta X\right]_0^\pi \ . \tag{1.7}$$

The first term gives the equation of motion which is the wave equation

$$\frac{\partial^2 X^\mu}{\partial\tau^2} - \frac{\partial^2 X^\mu}{\partial\sigma^2} = 0 \ . \tag{1.8}$$

In order to obtain this equation, the two other terms must vanish. This is done for the former by the conservation of momentum. For the latter we have to impose boundary conditions and thus distinguish the closed string from the open one.

- **Closed string.** In this case, the functions X^μ are periodic in σ

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi) \ . \tag{1.9}$$

- **Open string.** For the open string, there are two different choices.
  - Neumann boundary conditions. The ends of the string are free :

$$\frac{\partial X^\mu}{\partial \sigma}(\sigma = 0) = \frac{\partial X^\mu}{\partial \sigma}(\sigma = \pi) = 0 \ . \tag{1.10}$$

  - Dirichlet boundary conditions. The ends of the string are fixed :

$$X^\mu(\sigma = 0) = X^\mu_0 \quad \text{and} \quad X^\mu(\sigma = \pi) = X^\mu_\pi \ . \tag{1.11}$$

From now on we will focus only on the closed string. The equation of motion can be solved introducing the world-sheet light-cone coordinates

$$\sigma^\pm = \tau \pm \sigma \ . \tag{1.12}$$

The equation of motion becomes

$$\partial_+ \partial_- X^\mu = 0 \ , \tag{1.13}$$

which admits

$$X^\mu(\sigma, \tau) = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+) \tag{1.14}$$

as a solution. Here the functions X_R and X_L stand for the right- and left-moving parts of the string.

The closed string solution can be expanded in Fourier modes :

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2\, p^\mu \sigma^- + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n \, e^{-2in\sigma^-} \ , \tag{1.15}$$

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2\, p^\mu \sigma^+ + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n \, e^{-2in\sigma^+} \ , \tag{1.16}$$

where x^μ is the center-of-mass position, $p^\mu = \frac{1}{l_s}\alpha^\mu_0$ is the total string momentum and l_s is the string length scale, related to the string tension and the slope parameter α' by

$$T = \frac{1}{2\pi\alpha'} \quad \text{and} \quad \frac{1}{2}l_s^2 = \alpha' \ . \tag{1.17}$$

Finally the solution is

$$X^\mu = \frac{1}{2} l_s^2\, p^\mu \tau + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n} \left(\alpha^\mu_n \, e^{-2in\sigma^-} + \tilde{\alpha}^\mu_n \, e^{-2in\sigma^+} \right) \ . \tag{1.18}$$

In order for the function X^μ to be real, x^μ and p^μ must be real and the Fourier modes α^μ_n and $\tilde{\alpha}^\mu_n$ have to be conjugate to each other

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^* \quad \text{and} \quad \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^* \ . \tag{1.19}$$

We also have to impose the equation of motion for the metric. That is the cancellation of the energy-momentum tensor, which leads to the Virasoro constraints :

$$\left(\dot{X} \pm X' \right)^2 = 0 \ . \tag{1.20}$$

Poisson brackets

In order to anticipate the quantization of the string, we write the classical Poisson brackets for the dynamical variables :

$$[P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')]_{P.B.} = [X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')]_{P.B.} = 0 \ , \tag{1.21}$$

$$[P^\mu(\tau,\sigma), X^\nu(\tau,\sigma')]_{P.B.} = \delta(\sigma-\sigma')\,\eta^{\mu\nu} \ , \tag{1.22}$$

where $P^\mu = \frac{\delta S}{\delta \dot{X}_\mu} = T\dot{X}^\mu$ is the canonical momentum. For the Fourier modes the Poisson brackets become

$$[\alpha^\mu_m(\tau,\sigma), \alpha^\nu_n(\tau,\sigma')]_{P.B.} = [\tilde{\alpha}^\mu_m(\tau,\sigma), \tilde{\alpha}^\nu_n(\tau,\sigma')]_{P.B.} = -i\,m\,\eta^{\mu\nu}\,\delta_{m+n,0} \ , \tag{1.23}$$

$$[\alpha^\mu_m(\tau,\sigma), \tilde{\alpha}^\nu_n(\tau,\sigma')]_{P.B.} = 0 \ , \tag{1.24}$$

$$[x^\mu, p^\nu]_{P.B.} = \eta^{\mu\nu} \ . \tag{1.25}$$

1.1.2 Quantization of the bosonic string

Light-cone gauge quantization

In order to perform the quantization of the closed string, we will place ourselves in the framework of the light-cone gauge by first introducing the light-cone coordinates for space-time

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right) \ . \tag{1.26}$$

The $D-2$ remaining transverse coordinates are noted X^I.

Then in the conformal gauge a residual gauge symmetry still remains :

$$\sigma^\pm \longrightarrow \xi^\pm(\sigma^\pm) \ , \tag{1.27}$$

which can be used to make the choice of the light-cone gauge where

$$X^+(\tilde{\tau},\tilde{\sigma}) = x^+ + l_s^2\,p^+\,\tilde{\tau} \ . \tag{1.28}$$

This corresponds to setting

$$\alpha^+_n = 0 \ \text{ and } \ \tilde{\alpha}^+_n = 0 \quad \text{for} \quad n \neq 0 \ . \tag{1.29}$$

The variable X^- can be expressed in terms of X^I using the Virasoro constraints (1.20) :

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2\,p^+\,l_s^2}(\dot{X}^I \pm X^{I\prime})^2 \ . \tag{1.30}$$

Thus in the light-cone gauge both X^+ and X^- can be eliminated and only the transverse oscillators X^I remain.

The quantization of the closed string in the light-cone gauge is now within reach. Replacing fields by operators, Poisson brackets by commutators and making the substitution

$$[f,g]_{P.B.} \longrightarrow i\,[\hat{f},\hat{g}] \ , \tag{1.31}$$

give

$$[\alpha_m^I(\tau,\sigma),\alpha_n^J(\tau,\sigma')] = [\tilde{\alpha}_m^I(\tau,\sigma),\tilde{\alpha}_n^J(\tau,\sigma')] = m\,\eta^{IJ}\,\delta_{m+n,0}\ , \tag{1.32}$$

$$\left[\alpha_m^I(\tau,\sigma),\tilde{\alpha}_n^J(\tau,\sigma')\right] = 0\ , \tag{1.33}$$

$$[x^\mu,p^\nu] = i\,\eta^{\mu\nu}\ . \tag{1.34}$$

So now the condition (1.19) becomes

$$(\alpha_m^I)^\dagger = \alpha_{-m}^I \ \text{ and } \ (\tilde{\alpha}_m^I)^\dagger = \tilde{\alpha}_{-m}^I\ . \tag{1.35}$$

The ground state $|0,p^\mu\rangle$, of momentum p^μ, is defined as the state which is annihilated by all the lowering operators :

$$a_n^I\,|0,p^\mu\rangle = \tilde{a}_n^J\,|0,p^\mu\rangle = 0\ , \tag{1.36}$$

where $a_m^I = \frac{1}{\sqrt{m}}\alpha_m^I$ and $\tilde{a}_m^I = \frac{1}{\sqrt{m}}\tilde{\alpha}_m^I$. Then the other states $|\Phi\rangle$ are obtained by applying various times the creation operator on the ground state :

$$|\Phi\rangle = a_{n_1}^{I_1\,\dagger}\cdots a_{n_l}^{I_l\,\dagger}\,\tilde{a}_{n_1}^{J_1\,\dagger}\cdots\tilde{a}_{n_l}^{J_l\,\dagger}\,|0,p^\mu\rangle\ . \tag{1.37}$$

Mass-shell condition and spectrum

In this section we will use the main results of the Virasoro algebra theory, without explaining it in detail, as our goal is only to determine the mass-shell condition and the closed string spectrum.

The Virasoro operators are defined as the Fourier modes of the energy-momentum tensor :

$$T_{++} = 2\,l_s^2\sum_{n=-\infty}^{+\infty}\tilde{L}_m\,e^{-2im(\tau+\sigma)} \ \text{ and } \ T_{--} = 2\,l_s^2\sum_{n=-\infty}^{+\infty}L_m\,e^{-2im(\tau-\sigma)}\ , \tag{1.38}$$

where

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\alpha_{m-n}\cdot\alpha_n \ \text{ and } \ \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n\ . \tag{1.39}$$

In particular

$$H = L_0+\tilde{L}_0 \ \text{ and } \ P = L_0-\tilde{L}_0\ . \tag{1.40}$$

These are the classical expression of the Virasoro generators. The Virasoro constraints (1.20) simply become

$$L_m = \tilde{L}_m = 0\ . \tag{1.41}$$

In the quantum theory, the normal ordering must be taking into account. Thus, the Virasoro operators become :

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty}:\alpha_{m-n}\cdot\alpha_n: \ \text{ and } \ \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty}:\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n:\ . \tag{1.42}$$

Here the colons stand for the normal-ordering prescription where the lowering operators always appear to the right. They verify the algebra

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ , \tag{1.43}$$

where the second term on the right is called the conformal anomaly and $c = D$ is called the central charge.
In particular, L_0 becomes

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n \quad . \tag{1.44}$$

It can be rewritten as

$$L_0 = L_{0,cl} - a \ , \tag{1.45}$$

where $L_{0,cl}$ corresponds to the classical case and $a = \frac{D-2}{2} \sum_{n=1}^{+\infty} n$ is a normal-ordering constant. It is *a priori* infinite, but we will see later that we can give it a finite value. Then to obtain the quantum theory, we will just replace L_0 by $L_0 - a$. Of course we can derive the same formulae for $\tilde{L}_0$.

From (1.41), we get that the physical states $|\Phi\rangle$ are described by the conditions

$$(L_0 - a)\,|\Phi\rangle = (\tilde{L}_0 - a)\,|\Phi\rangle = 0 \ , \tag{1.46}$$

$$L_m\,|\Phi\rangle = \tilde{L}_m\,|\Phi\rangle = 0 \ \text{ for } m > 0 \ . \tag{1.47}$$

The mass-shell condition in the light-cone gauge can now be determined using the previous equations :

$$M^2 = -p^\mu\, p_\mu = \frac{4}{\alpha'}\,(N-a) \ , \tag{1.48}$$

where

$$N = \sum_{i=1}^{D-2} \sum_{n>0} \alpha^I_{-n} \alpha^I_n \ ,$$

is the number operator. Similarly taking $\tilde{N} = \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}^I_{-n} \tilde{\alpha}^I_n$, we get

$$M^2 = \frac{1}{\alpha'}\left(\tilde{N} - a\right) \ . \tag{1.49}$$

The difference between the equations (1.48) and (1.49) gives the level-matching condition

$$N = \tilde{N} \ , \tag{1.50}$$

which means that the number of right-moving oscillators must be equal to the number of left-moving oscillators. The mass-shell condition can thus be rewritten as

$$M^2 = \frac{2}{\alpha'}\left(N + \tilde{N} - 2a\right) \ . \tag{1.51}$$

We now turn to the spectrum by focusing on the states of lowest energy. Indeed, the states with a mass at the string level would have a mass of the order of the Planck mass and thus are not phenomenologically acceptable.

- **The ground state $|\mathbf{0}, \mathbf{p}^\mu\rangle$.** The mass-shell equation condition is $M^2 = -\frac{a}{\alpha'}$. We will see later that $a = 1$, so this state has a negative mass and is a tachyon.

- **The first excited level $\alpha^{\mathbf{I}}_{\mathbf{-1}}\tilde{\alpha}^{\mathbf{J}}_{\mathbf{-1}}\,|\mathbf{0}, \mathbf{p}^\mu\rangle$.** This state corresponds to $N = 1$, such that $M^2 = \frac{1}{\alpha'}\,(1-a)$. It belongs to a representation of $SO(D-2)$. Lorentz invariance requires that a physical states belongs to a representation of $SO(D-1)$ for massive states and of $SO(D-2)$ for massless states. As a consequence, the first excited state must be massless and $a = 1$. Using ζ-function regularization, it can be given a finite value to the normal-ordering constant a :

$$a = \frac{D-2}{24}\ , \tag{1.52}$$

  which implies that $D = 26$.

  The first excited level can be split into a symmetric and traceless part corresponding to the graviton field, a trace term which is the scalar dilaton field and an antisymmetric second-rank tensor field.

- Then, according to the mass-shell condition, the next states (with $N > 1$) will all be massive.

The normal-ordering constant a and the dimension D of the space-time has been determined using Lorentz invariance. However, there exists other possibilities to determine these constants, *e.g.* the conformal invariance.

Finally, we have seen that the bosonic string requires a critical space-time dimension $D = 26$, with the existence of the tachyon, which must be avoided. Moreover, it is a theory with only bosons. In order to obtain fermions and no tachyon, introducing the superstring is necessary. It will be presented in the next part.

1.2 The superstring

1.2.1 The Ramond-Neveu-Schwartz formalism

The superstring action

The world-sheet functions $X^\mu(\tau,\sigma)$ describe the bosonic string. In order to add fermionic fields to our theory, we introduce world-sheet variables $\psi^\mu(\tau,\sigma)$. They describe two-components Majorana spinors which are anticommuting, in agreement with the spin statistics :

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \tag{1.53}$$

$$\{\psi^\mu_A(\tau,\sigma), \psi^\nu_B(\tau,\sigma')\} = \pi\,\delta_{AB}\,\delta(\sigma-\sigma')\,\eta^{\mu\nu} \tag{1.54}$$

The superstring action is obtained by adding the standard Dirac action for fermions to the bosonic string action in conformal gauge,

$$S = -\frac{T}{2}\int d\tau d\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \ , \tag{1.55}$$

where the ρ^α are the two-dimensional Dirac matrices, with $\alpha = 0, 1$

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{1.56}$$

which obey the anticommutation relation $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$. The fermionic part of the action can be rewritten

$$S_f = iT \int d^2\sigma \left(\psi_-^\mu \partial_+ \psi_{-\mu} + \psi_+^\mu \partial_- \psi_{+\mu}\right) \ , \tag{1.57}$$

where we use the light-cone coordinates σ_+ and σ_-.

As previously a conserved supersymmetric current J^μ and an energy-momentum tensor $T^{\mu\nu}$ can be defined using the Noether procedure for the world-sheet supersymmetric and translational invariances. So the Virasoro constraints (1.20) become

$$J_+ = J_- = T_{++} = T_{--} = 0 \ . \tag{1.58}$$

Boundary conditions

The boundary conditions and the Fourier expansion for the bosonic fields X^μ are the same as those described in the previous part on the bosonic string. The variation of the fermionic action (1.57) is

$$\delta S_f = -T \int d\tau d\sigma \left(\delta\psi_-^\mu \partial_+ \psi_{-\mu} + \delta\psi_+^\mu \partial_- \psi_{+\mu}\right) - \frac{T}{2}\int d\tau \left[\psi_+^\mu \delta\psi_{+\mu} - \psi_-^\mu \delta\psi_{-\mu}\right]_{\sigma=0}^{\sigma=\pi} \ . \tag{1.59}$$

So the equations of motion are

$$\partial_+ \psi_- = 0 \ \text{ and } \ \partial_- \psi_+ = 0 \tag{1.60}$$

These are the Weyl conditions for spinors in two-dimensions. The fields ψ_- and ψ_+ are thus Majorana-Weyl spinors. These equations imply that

$$\psi_-^\mu = \psi_-^\mu(\tau - \sigma) \ \text{ and } \ \psi_+^\mu = \psi_+^\mu(\tau + \sigma) \ . \tag{1.61}$$

As previously we require the cancellation of the second term in (1.59),

$$\psi_+^\mu(\tau, \sigma_*)\, \delta\psi_{+\mu}(\tau, \sigma_*) - \psi_-^\mu(\tau, \sigma_*)\, \delta\psi_{-\mu}(\tau, \sigma_*) = 0 \ , \tag{1.62}$$

at each endpoint $\sigma_* = 0$ or $\sigma_* = \pi$. This will give the boudary conditions for the superstring.

We now consider the closed superstring, for which we impose periodic boundary conditions. In the bosonic case, the closed string corresponds to the tensor product of a left-moving and a right-moving string. It is the same here so periodic or antiperiodic boundary conditions can be imposed separately for the right- and left-movers :

$$\psi^\mu_-(\tau,\sigma) = \pm\,\psi^\mu_-(\tau,\sigma+\pi)\ , \tag{1.63}$$
$$\psi^\mu_+(\tau,\sigma) = \pm\,\psi^\mu_+(\tau,\sigma+\pi)\ . \tag{1.64}$$

The plus sign corresponds to the Ramond (R) boundary conditions, which are periodic, while the minus sign corresponds to the Neveu-Schwartz (NS) boundary conditions, which are antiperiodic. Therefore for the closed superstring the possible sectors are NS-NS, R-R, NS-R or R-NS, where the left-movers are written on the left.

The mode expansion for the functions $\psi^\mu_\pm$ are :

– **Ramond boundary conditions :**

$$\psi^\mu_-(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in(\tau-\sigma)}\ , \tag{1.65}$$
$$\psi^\mu_+(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{d}^\mu_n e^{-2in(\tau+\sigma)}\ . \tag{1.66}$$

– **Neveu-Schwartz boundary conditions :**

$$\psi^\mu_-(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} b^\mu_r e^{-2ir(\tau-\sigma)}\ , \tag{1.67}$$
$$\psi^\mu_+(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}^\mu_r e^{-2ir(\tau+\sigma)}\ . \tag{1.68}$$

1.2.2 Quantization of the superstring

Quantization of the closed superstring

The closed superstring quantization can now be achieved. The method is the same as previously. In the superstring theory, the light-cone gauge can still be defined, by setting

$$X^+(\tau,\sigma) = x^+ + p^+\tau\ , \tag{1.69}$$
$$\psi^+(\tau,\sigma) = 0\ . \tag{1.70}$$

Because of the Virasoro constraints (1.58), X^- and ψ^- can be expressed as functions of the transverse fermionic and bosonic oscillators.

For the bosonic part X^I, the commutation relations are the same than for the bosonic string :

$$[\alpha^I_m,\alpha^J_n] = [\tilde{\alpha}^I_m,\tilde{\alpha}^J_n] = m\,\delta_{m+n,0}\,\eta^{IJ}\ , \tag{1.71}$$

and all other commutators vanish.

For the fermionic part, the anticommutation relations are :

$$\{b_r^I, b_s^J\} = \{\tilde{b}_r^I, \tilde{b}_s^J\} = \delta_{r+s,0}\, \eta^{IJ} \ , \tag{1.72}$$

$$\{d_m^I, d_n^J\} = \{\tilde{d}_m^I, \tilde{d}_n^J\} = \delta_{m+n,0}\, \eta^{IJ} \ , \tag{1.73}$$

and all other anticommutators vanish.

In both sectors, the ground state $|0, p^\mu\rangle_{NS/R}$ of momentum p^μ is defined as the state which is annihilated by all the lowering operators :

$$\alpha_m^I\, |0, p^\mu\rangle_{NS} = b_r^J\, |0, p^\mu\rangle_{NS} = 0 \qquad \text{for} \qquad m, r > 0 \ , \tag{1.74}$$

$$\alpha_m^I\, |0, p^\mu\rangle_R = d_m^K\, |0, p^\mu\rangle_R = 0 \qquad \text{for} \qquad m > 0 \ , \tag{1.75}$$

and similarly for the left-moving modes.

In the Neveu-Schwartz sector the ground state is unique, whereas in the Ramond sector it is degenerate. Actually the operators d_0^I can act on it without changing the mass, as we will see later.

Then the excited states are obtained by acting on the ground state with the creation operators α_{-m}^I, $\tilde{\alpha}_{-m}^I$, b_{-r}^I and $\tilde{b}_{-r}^I$ for the Neveu-Schwartz sector and α_{-m}^I, $\tilde{\alpha}_{-m}^I$, d_{-n}^I and $\tilde{d}_{-n}^I$ for the Ramond sector.

Super-Virasoro modes and mass-shell condition

The super-Virasoro generators are defined as the Fourier modes of the energy-momentum tensor :

$$L_m = L_m^{(b)} + L_m^{(f)} \ , \tag{1.76}$$

where

$$L_m^{(b)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \ :\ \alpha_{-n} \cdot \alpha_{m+n} \ :\ , \tag{1.77}$$

and

$$L_m^{(f)} = \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} \left(r + \frac{m}{2} \right) \ :\ b_{-r} \cdot b_{m+r} \ :\quad \text{for the NS sector,} \tag{1.78}$$

$$L_m^{(f)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(n + \frac{m}{2} \right) \ :\ d_{-n} \cdot d_{m+n} \ :\quad \text{for the R sector.} \tag{1.79}$$

The Fourier modes of the supercurrents are defined as:

$$G_r = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot b_{r+n} \ \ , \ r \in \mathbb{Z} + 1/2 \ \ , \text{for the NS sector,} \tag{1.80}$$

$$F_m = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot d_{m+n} \ \ , \ m \in \mathbb{Z} \ \ , \text{for the R sector.} \tag{1.81}$$

There are similar equations for the left-moving modes.

The spectrum of the closed superstring can now be determined. The physical states conditions are :

– **NS sector :**

$$L_m |\Phi\rangle = \tilde{L}_m |\Phi\rangle = 0 \qquad \text{for } m > 0 \ , \tag{1.82}$$
$$G_r |\Phi\rangle = \tilde{G}_r |\Phi\rangle = 0 \qquad \text{for } r > 0 \ , \tag{1.83}$$
$$(L_0 - a_{NS}) |\Phi\rangle = (\tilde{L}_0 - a_{NS}) |\Phi\rangle = 0 \ . \tag{1.84}$$

– **R sector :**

$$L_m |\Phi\rangle = \tilde{L}_m |\Phi\rangle = 0 \qquad \text{for } m > 0 \ , \tag{1.85}$$
$$F_n |\Phi\rangle = \tilde{F}_n |\Phi\rangle = 0 \qquad \text{for } n \geq 0 \ , \tag{1.86}$$
$$(L_0 - a_R) |\Phi\rangle = (\tilde{L}_0 - a_R) |\Phi\rangle = 0 \ . \tag{1.87}$$

The massless condition becomes :

$$M^2 = \frac{2}{\alpha'} \left(N_A + \tilde{N}_B - a_A - a_B \right) \ , \tag{1.88}$$

where A and B stands for the sector (NS or R). N and $\tilde{N}$ are the number operators and are defined by :

$$N_{NS} = \sum_{n=1}^{+\infty} \alpha^i_{-n} \alpha^i_n + \sum_{r=1/2}^{+\infty} r\, b^i_{-r} b^i_r \ , \tag{1.89}$$

$$N_R = \sum_{n=1}^{+\infty} \alpha^i_{-n} \alpha^i_n + \sum_{n=1/2}^{+\infty} n\, d^i_{-n} d^i_n \ , \tag{1.90}$$

and similarly for the left-moving number operators.
The level-matching condition (1.50) becomes :

$$N_A - a_A = \tilde{N}_B - a_B \tag{1.91}$$

As previously the normal-ordering constant can be determined using Lorentz invariance. It gives :

$$a_{NS} = \frac{1}{2} \ \text{ and } \ a_R = 0 \ . \tag{1.92}$$

In particular this implies that the critical dimension for the superstring is

$$D = 10 \ . \tag{1.93}$$

Superstring massless spectrum

The construction of the massless superstring spectrum is achieved in two phases. Since the closed superstring is a tensor product of right- and left-moving, which could each be in the Ramond or Neveu-Scwartz sector, we will first determine the possible states in both sectors. Then we will proceed to the GSO projection in order to get rid of the tachyonic state.

The Neveu Schwartz sector

- The ground state $|0, p^\mu\rangle_{NS}$ mass is given by $\alpha' M^2 = -\frac{1}{2}$. It is the tachyon, which must be avoided. From now on it will be noted it $|NS\rangle$.

- The first excited state is constructed by acting with the raising operators having the smallest associated frequency : $b^i_{-1/2}\,|NS\rangle$. It is a massless vector with eight transverse components.

The Ramond sector

- The ground state $|0, p^\mu\rangle_R$ is a degenerate state. Indeed, as discussed above, acting on it with the oscillators d^i_0 does not change the mass. As the zero modes obey a Clifford algebra, they can be represented by Dirac matrices. Using simple linear combinations of them they can be divided into raising and lowering operators. The resulting states, which are constructed by acting with these operators in all possible way, fall into a $(2^{8/2} = 16)$-dimensional spinorial representation. By further considerations, we can show that these states can be split into two different representations **8** of $SO(8)$, whether they have an even or odd number of oscillators acting on them. The Ramond ground states, which are now called $|R\pm\rangle$, are massless.

- The first excited state is constructed by acting with the raising operators α^i_{-1} and d^i_{-1} on the Ramond ground state. They acquire masses, so they are not of interest.

The GSO projection. The last stage to find the spectrum of the closed superstring is to apply the GSO projection on these states. This method was first introduced by Gliozzi, Scherk and Olive.

First, we need to define the G-parity operator in the Neveu-Schwartz sector as :

$$G_{NS} = (-1)^{F_{NS}+1} \;, \tag{1.94}$$

where $F_{NS} = \sum_{r=1/2}^{+\infty} b^i_{-r} b^i_r$ is the number of b-oscillator excitations. Thus, the operator determines whether a state has an even or odd number of world-sheet fermions oscillations.

In the Ramond sector, this oscillator is given by :

$$G_R = \Gamma_{11}(-1)^{F_R} \;, \tag{1.95}$$

where $F_R = \sum_{n=1}^{+\infty} d^i_{-n} d^i_n$ is the fermion number operator in the Ramond sector and $\Gamma_{11} = \Gamma_0\Gamma_{11}\ldots\Gamma_9$ is the ten dimensional analog of the chirality matrice γ_5 in four dimensions. The G-parity operator in the Ramond sector projects the states on the representation with either an even or an odd number of oscillators.

The GSO projection consists in keeping only the states with a positive G-parity in the Neveu-Schwartz sector. As a consequence the tachyon will be projected out.

In the Ramond sector, states with positive or negative G-parity can be kept since the ground state is massless.

Then the ground state in the Neveu-Schwartz sector is an eight-componant vector boson, which matches with the fact that the ground state in the Ramond sector is an eight-componant spinor. This is an indication of the spectrum space-time supersymmetry. Actually at each mass level there is an equal number of bosons and fermions, as required by supersymmetry.

The closed superstring massless spectrum can now be explicited. There are two possibilities depending on whether we choose for the left- and right-moving Ramond sector a different parity or not.

Type IIA closed superstring :

$$|R-\rangle \otimes |R+\rangle \ , \tag{1.96}$$

$$\tilde{b}^i_{-1/2}\ |NS+\rangle \otimes b^j_{-1/2}\ |NS+\rangle \ , \tag{1.97}$$

$$\tilde{b}^i_{-1/2}\ |NS+\rangle \otimes |R+\rangle \ , \tag{1.98}$$

$$|R-\rangle \otimes b^i_{-1/2}\ |NS+\rangle \ . \tag{1.99}$$

Type IIB closed superstring :

$$|R+\rangle \otimes |R+\rangle \ , \tag{1.100}$$

$$\tilde{b}^i_{-1/2}\ |NS+\rangle \otimes b^j_{-1/2}\ |NS+\rangle \ , \tag{1.101}$$

$$\tilde{b}^i_{-1/2}\ |NS+\rangle \otimes |R+\rangle \ , \tag{1.102}$$

$$|R+\rangle \otimes b^i_{-1/2}\ |NS+\rangle \ . \tag{1.103}$$

A few observations can be made on this spectrum :

– The first sector contains bosons with either the same or opposite chirality.

– In both cases, the second sector $\tilde{b}^i_{-1/2}\ |NS+\rangle \otimes b^j_{-1/2}\ |NS+\rangle$ contains the graviton, the dilaton and the second-rank antisymetric tensor. These states are bosons.

– Each of the two last sectors contains the spin $\frac{3}{2}$ gravitino and the spin $\frac{1}{2}$ dilatino, superpartners of the graviton and dilaton. Therefore the superstring has a $\mathcal{N} = 2$ supersymmetry coming from the left- and right-moving sectors.

We finally end up with a ten-dimensional superstring theory with $\mathcal{N} = 2$ supersymmetry and without tachyon.

1.3 The heterotic string

We would prefer to have a type of superstring, namely the heterotic string, which has four dimensions and a $\mathcal{N} = 1$ supersymmetry. It is the framework used in our work so we will now introduce it.

1.3.1 Construction of the heterotic string

In order to obtain $\mathcal{N} = 1$ supersymmetry on the superstring, we decouple the left- and right-moving modes. The supersymetric charges are carried by the left-moving currents, while the right-moving world-sheet fields are described by the formalism of the bosonic string.

The construction of the heterotic superstring occurs as follows.

For the left-moving sector, we consider the left-moving superstring fields X^μ_+ and ψ^μ_+ with $\mu = 0, \ldots, 9$. The critical dimension is $D = 10$.

Then for the right-moving sector we have the ten bosonic right-movers X^μ_-. Since a space-time boson contributes a unit to the central charge and a free-fermion contributes half a unit, 32 Majorana-Weyl right-moving free-fermions λ^i_- are needed to cancel the conformal anomaly $c = -26$ in the bosonic string.

The theory is still ten-dimensional because the space-time indices $\mu = 0, \ldots, 9$ are carried by the coordinates X^μ in both right- and left-moving sectors, while the internal fermions λ^i_- do not carry a space-time indice.

To sum up, we have to consider the following fields :

$$X^\mu_+ \;\text{ and }\; \psi^\mu_+ \;\text{ in the left-moving sector,} \tag{1.104}$$

$$X^\mu_- \;\text{ and }\; \lambda^i_- \;\text{ in the right-moving sector,} \tag{1.105}$$

where $\mu = 0, \ldots, 9$ and $i = 1, \ldots, 32$.

The action for the heterotic string is thus

$$S = \frac{1}{\pi} \int d^2\sigma \left(2\partial_- X_\mu \partial_+ X^\mu + i\psi^\mu \partial_- \psi_\mu + i \sum_{i=1}^{32} \lambda^i \partial_+ \lambda^i \right) . \tag{1.106}$$

1.3.2 Compactification to four dimensions

We are now looking for a four-dimensional theory. This can be achieved by following the same procedure as previously.

Considering a four-dimensional space-time, that is the indice μ now runs from 0 to 3, 18 Majorana-Weyl fermionic left-moving fields λ^i and 12 Majorana-Weyl fermionic right-moving fields λ^j are needed to cancel the conformal anomaly in both sectors.

The total set of fields is now

$$X_+^\mu \ \ , \ \psi_+^\mu \ \text{ and } \ \lambda_+^j \ \text{ in the left-moving sector,} \tag{1.107}$$

$$X_-^\mu \ \text{ and } \ \lambda_-^i \ \text{ in the right-moving sector,} \tag{1.108}$$

where $\mu = 0, \ldots, 9$, $i = 1, \ldots, 44$ and $j = 1, \ldots, 18$.

Let us adopt the complex coordinates, defining

$$z = \tau + i\sigma \ \text{ and } \ \bar{z} = \tau - i\sigma \ . \tag{1.109}$$

The world-sheet fields are now functions of z and $\bar{z}$. From now on the following notations will be used for the fields :

$$X^\mu(z, \bar{z}) \ , \ \mu = 1, 2 \ , \tag{1.110}$$

$$\psi^\mu(z) \ , \ \mu = 1, 2 \ , \tag{1.111}$$

$$\lambda^i(z) \ , \ i = 1, \ldots, 18 \ , \tag{1.112}$$

$$\bar{\lambda}^j(\bar{z}) \ , \ j = 1, \ldots, 44 \ . \tag{1.113}$$

We have placed ourselves in the light-cone gauge, such that the space-time bosons and fermions have only two degrees of freedom, namely the transverse coordinates.

The heterotic action can be rewritten as

$$S = \frac{1}{\pi} \int d^2z \left(\partial_z X_\mu \partial_{\bar{z}} X^\mu - 2i\psi^\mu \partial_z \psi_\mu - 2i \sum_{i=1}^{18} \lambda^i \partial_z \lambda^i - 2i \sum_{j=1}^{44} \bar{\lambda}^j \partial_{\bar{z}} \bar{\lambda}^j \right) \ . \tag{1.114}$$

The heterotic string theory we have just described is the free-fermionic formulation. It will be the framework of the rest of this report.

Part 2

Free-fermionic models

Our goal is the construction of realistic models in the heterotic string theory. In this part we describe the free fermionic formalism derived by Antoniadis, Bacchas and Kounnas in [1, 2] and by Kawai, Lewellen and Tye in [13]. We explain how the modular invariance constraint of the one-loop partition function leads to a set of rules which will be our tool for constructing new free fermionic models. The method for constructing string vacua and extracting the massless spectrum is then summarized and applied to a first example.

2.1 Fermionic formalism and partition function

2.1.1 Fermionic formalism of the heterotic string

We place ourselves in the framework of the heterotic string theory. As seen in the previous part, we can interpret the degrees of freedom needed to cancel the conformal anomaly as free fermions propagating on the string world-sheet. To formulate the theory directly in 4-dimensions we need 18 left-moving real Majorana-fermions and 44 right-moving real Majorana-Weyl fermions.

In the light-cone gauge we have the following world-sheet field content :

Left-moving	$X_L^\mu(z)$	$\mu = 1, 2$	2 transverse coordinates
	$\psi_L^\mu(z)$	$\mu = 1, 2$	their superpartners
	$\chi_I(z), y_I(z), w_I(z)$	$I = 1 \ldots 6$	18 internal real fermions
Right-moving	$X_R^\mu(\bar{z})$	$\mu = 1, 2$	2 transverse coordinates
	$\bar{\lambda}^i(\bar{z})$	$i = 1 \ldots 44$	44 internal real fermions

In the Polyakov picture string theory is formulated as a perturbative sum over path integral on the string world-sheet, a genus-g Riemann surface. The free fermions can be propagated around non-contractible loops on this surface so we will have to specify boundary conditions for each world-sheet fermion.

World-sheet supersymmetry should be preserved, which imposes that the supercurrent T_F must be uniquely defined, up to a sign, under the transformation of the world-sheet fermions. The supercurrent is taken to be

$$T_F = \psi^\mu \partial X_\mu + i \sum_I \chi^I y^I w^I, \tag{2.1}$$

where the $\{\chi_I(z), y_I(z), w_I(z)\}$ transform as the adjoint representation of $SU(2)^6$. The transport properties of right-moving fermions and left-moving fermions (for which we need to consider the transformation of the supercurrent) around a non-contractible loop show that we can realise any configuration of boundary conditions in some basis consisting of 64 fermions.

We note here that we can pair two real fermions into a complex fermion in the following way :

$$\lambda^{kl} = \frac{1}{\sqrt{2}}(\lambda^k + i\lambda^l). \tag{2.2}$$

In this report we will use the following notations for the basis fermions :
Real left fermions : $\{\psi^{\mu 1}, \psi^{\mu 2}, \chi^1, y^1, w^1, \chi^2, y^2, w^2, \chi^3, y^3, w^3, \chi^4, y^4, w^4, \chi^5, y^5, w^5, \chi^6, y^6, w^6\}$
Real right fermions : $\{\bar{y}^1, \bar{w}^1, \bar{y}^2, \bar{w}^2, \bar{y}^3, \bar{w}^3, \bar{y}^4, \bar{w}^4, \bar{y}^5, \bar{w}^5, \bar{y}^6, \bar{w}^6\}$
Complex right fermions : $\{\bar{\psi}^1, \bar{\psi}^2, \bar{\psi}^3, \bar{\psi}^4, \bar{\psi}^5, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3, \bar{\Phi}^1, \bar{\Phi}^2, \bar{\Phi}^3, \bar{\Phi}^4, \bar{\Phi}^5, \bar{\Phi}^6, \bar{\Phi}^7, \bar{\Phi}^8\}$

We will sometimes complexify some of the left-moving fermions and use the following complex fermions : $\{\psi^\mu, \chi^{12}, \chi^{34}, \chi^{56}\}$.

2.1.2 String amplitude on the torus

String amplitude

In the Polyakov approach to string theory, a string amplitude is calculated by the path integral

$$A_n = \sum_{g=0}^{\infty} \int DhDX^\mu \int \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_n V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n), \tag{2.3}$$

where we must insure that we sum only over physically inequivalent paths. The V_i are vertex operators of external string states on the genus-g Riemann surface defined by the world-sheet.

Therefore the total string amplitude is a sum over all possible Riemann surfaces moded out by conformal invariance, similar to the sum over all Feynman graphs in Quantum Field Theory. The conformal invariance maps the tree level string topology to the sphere and the one-loop topology to the torus.

At tree level all reparametrizations are local and quantum corrections are not taken into account, but at higher level further constraints will arise. Hence it is instructive to look at the one-loop vacuum to vacuum amplitude, with no external states. This is the one-loop partition function.

Torus and modular invariance

The one-loop string amplitude is a sum over all non-equivalent tori. We have to look at the symmetries of a torus to determine what are the non-equivalent tori.

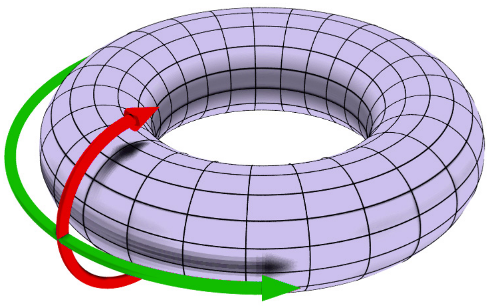

Figure 2.1: The two non-contractible loops of the torus.

We can map the torus to the complex plane by cutting it along its two non contractible loops shown on the Figure 2.1. The torus can be characterized by specifying two finite and nonzero periods in the complex plane λ_1, λ_2 with a nonreal ratio :

$$z \sim z + \lambda_1, \quad z \sim z + \lambda_2. \tag{2.4}$$

The torus is then identified with the conplex plane modulo a two-dimensional lattice $\Lambda_{(\lambda_1,\lambda_2)}$, where $\Lambda_{(\lambda_1,\lambda_2)} = \{m\lambda_1 + n\lambda_2,\ m, n \in \mathbb{Z}\}$.

Using the reparametrization $z \to z/\lambda_2$ the torus is equivalent to one whose periods are 1 and $\tau = \frac{\lambda_1}{\lambda_2}$, as shown in Figure 2.2.

The torus is left invariant by the two following transformations :

$$T : \tau \to \tau + 1 \text{ redefines the same torus,} \tag{2.5}$$

$$S : \tau \to -\frac{1}{\tau} \quad \text{swaps the two coordinates and reorients the torus.} \tag{2.6}$$

These transformations span a group of transformation, the modular group :

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, ab - cd = 1. \tag{2.7}$$

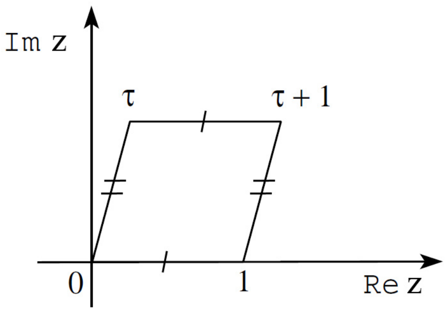

Figure 2.2: The torus mapped to the complex plan when the opposite edges of the parallelogram are identified.

Thus the modular group is $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, where the division by $\mathbb{Z}_2$ takes the equivalence of an $SL(2,\mathbb{Z})$ matrix and its negative into account. The moduli space $\mathscr{M}$ of the torus (the space of conformally inequivalent tori) is

$$\mathscr{M} \cong \mathbb{C}/PSL(2,\mathbb{Z}. \tag{2.8}$$

A fundamental domain can be taken as

$$\mathcal{F} = \{\tau||\tau| \geq 1, |Re\tau| \leq 1/2, Im\tau > 0\}. \tag{2.9}$$

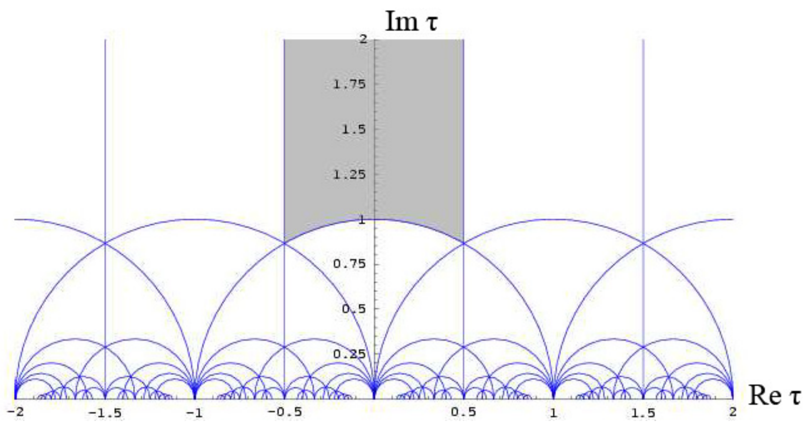

Figure 2.3: The shaded region is the fundamental region of the modular group of the torus.

So the partition function is a sum over this domain in order to integrate over all conformaly inequivalent tori. The $SL(2,\mathbb{Z})$ invariant measure over the fundamental domain is

$$\int \frac{d^2\tau}{(\text{Im}\tau)^2}.$$

In addition, we have to require that the partition function does not depend on the parametrization of the tori, so it should be invariant under the modular transformations spanned by T and S.

Boundary conditions

For each world-sheet fermion we have to specify boundary conditions that express the shift of phase of the fermionic field under parallel transport around each noncontractible loop of the torus :

$$f \to -e^{i\pi\alpha(f)} f. \tag{2.10}$$

For real fermions $\alpha(f)$ is either 0 (Neveu-Schwarz conditions) or 1 (Ramond conditions), whereas for complex fermions $\alpha(f) \in (-1, 1]$. Since there are two noncontractible loops on a torus, the complete phase assignment for a fermion can be written as a set of two phases

$$\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}. \tag{2.11}$$

A set of specified phases for all basis fermions for one noncontractible loop is called a spin-structure and is expressed as a 64-dimensional vector :

$$\alpha = \left\{ \alpha(\psi^{\mu 1}), \ldots, \alpha(\bar{\Phi}^8) \right\}. \tag{2.12}$$

The complete spin-structure assignment to all fermions on the torus can then be defined by two vectors

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{2.13}$$

2.1.3 The one-loop partition function

Now let us have a look at the partition function. One can think of the path integral on a torus of parameter $\tau = \tau_1 + i\tau_2$ as formed by a field on a circle that has been evolved for Euclidean time $2\pi\tau_2$, translated by $2\pi\tau_1$, and identified with the initial circle. The generator of translations in time is the Hamiltonian $H = L_0 + \bar{L}_0 + \frac{1}{24}$ and the generator of translation in space is the momentum operator $P = L_0 - \bar{L}_0$, as seen in equation (1.40). The identification of the ends of the cylinder thus formed is realized by taking the trace over the Hilbert space of states.

$$Z(\tau_1, \tau_2) = \sum_{s \in \mathcal{H}} \left\langle s \left| e^{2\pi i \tau_1 P} e^{-2\pi i \tau_2 H} \right| s \right\rangle \tag{2.14}$$

$$, = \operatorname*{Tr}_{\mathcal{H}} e^{2\pi i \tau_1 P} e^{-2\pi i \tau_2 H}, \tag{2.15}$$

which can be rewritten with $q \equiv e^{2\pi i \tau}$ and using (1.40) as

$$Z(\tau) = q^{-1/48} \bar{q}^{-1/48} \operatorname*{Tr}_{\mathcal{H}} q^{L_0} \bar{q}^{\bar{L}_0} \tag{2.16}$$

This can be calculated since we know how L_0 acts on the states space.

We can compute this for each fermion. If the time boundary condition is antiperiodic (NS), then the partition function is just given by the trace with L_0 acting on the appropriate R or NS Fock space :

$$Z_{NS}^{NS}(\tau) = \text{Tr}_{NS}\; q^{L_0 - 1/48} \quad \text{and} \quad Z_{R}^{NS}(\tau) = \text{Tr}_{R}\; q^{L_0 - 1/48}. \tag{2.17}$$

When the time boundary condition is periodic (R) the definition of the trace is modified:

$$Z_{NS}^{R}(\tau) = \text{Tr}_{NS}(-1)^F q^{L_0 - 1/48} \quad \text{and} \quad Z_{R}^{R}(\tau) = \text{Tr}_{R}(-1)^F q^{L_0 - 1/48}, \tag{2.18}$$

where F is the fermion number operator, defined by the relations

$$\begin{aligned} F(f) &= 1, \text{ if f is a fermionic oscillator,} & (2.19)\\ F(f) &= -1, \text{ if f is the complex conjugate of a fermionic oscillator,} & (2.20)\\ F\,|+\rangle_R &= 0, & (2.21)\\ F\,|-\rangle_R &= 1, & (2.22) \end{aligned}$$

where $|+\rangle_R$, $|-\rangle_R$ are the degenerated Ramond vacua.

The partition function must include all possible combinations of boundary conditions, it is therefore a sum over all spin-structures. All the previous work now leads to the complete partition function :

$$Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\text{Im}\tau)^2}\; Z_B^2 \sum_{\substack{\text{spin}\\ \text{structure}}} C\binom{\alpha}{\beta} \prod_{f=1}^{64} Z_F \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix} \tag{2.23}$$

Let us explain each term :

- $\frac{d\tau d\bar{\tau}}{(\text{Im}\tau)^2}$ is the invariant measure under the modular transformations of the torus.
- Z_B is the bosonic contribution

  $$Z_B = \frac{1}{\sqrt{|\text{Im}\tau|}\eta(\tau)},$$

  where

  $$\eta(\tau) = q^{\frac{1}{12}} \prod_n \left(1 - q^{2n}\right) \text{ with } q = e^{2\pi i \tau},$$

  is the Dedekind eta function.
- The $C\binom{\alpha}{\beta}$ are coefficients on the spin-structures that are yet to be determined.
- $Z_F \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}$ is the contribution of the fermion f, wich depends on its boundary conditions $\alpha(f)$ and $\beta(f)$. It can be calculated using (2.16) to obtain the following

results:

$$Z_F\begin{bmatrix}0\\0\end{bmatrix} = \sqrt{\frac{\vartheta_3}{\eta}}, \tag{2.24}$$

$$Z_F\begin{bmatrix}1\\0\end{bmatrix} = \sqrt{\frac{\vartheta_2}{\eta}}, \tag{2.25}$$

$$Z_F\begin{bmatrix}0\\1\end{bmatrix} = \sqrt{\frac{\vartheta_4}{\eta}}, \tag{2.26}$$

$$Z_F\begin{bmatrix}1\\1\end{bmatrix} = \sqrt{\frac{\vartheta_1}{\eta}}, \tag{2.27}$$

where ϑ_i are defined as

$$\vartheta_1 = \vartheta\begin{bmatrix}1\\1\end{bmatrix}, \quad \vartheta_2 = \vartheta\begin{bmatrix}1\\0\end{bmatrix}, \quad \vartheta_3 = \vartheta\begin{bmatrix}0\\0\end{bmatrix}, \quad \vartheta_4 = \vartheta\begin{bmatrix}0\\1\end{bmatrix}, \tag{2.28}$$

and

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix} = \sum_{n\in\mathbb{Z}} q^{\frac{(n-a/2)^2}{2}} e^{2\pi i(n-b/2)(n-a/2)}. \tag{2.29}$$

These formulae should be complex conjugated for the right moving fermions.

2.2 Model building and constraints

2.2.1 Modular invariance of the partition function

Let us recall the expression of the partition function. Its invariance under modular transformations will give us further constraints to build a model.

$$Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2}\, Z_B^2 \sum_{\substack{\text{spin}\\\text{structure}}} C\begin{pmatrix}\alpha\\\beta\end{pmatrix} \prod_{f=1}^{64} Z_F\begin{bmatrix}\alpha(f)\\\beta(f)\end{bmatrix} \tag{2.30}$$

The measure element and the bosonic contribution are modular invariant. Now we have to focus on the modular invariance of the other terms of the partition function.

Under $\tau \to \tau + 1$, we have the following transformations :

$$\eta \longrightarrow e^{i\pi/12}\eta, \tag{2.31}$$

$$\vartheta_1 \longrightarrow e^{i\pi/4}\vartheta_1, \tag{2.32}$$

$$\vartheta_2 \longrightarrow e^{i\pi/4}\vartheta_2, \tag{2.33}$$

$$\vartheta_3 \longleftrightarrow \vartheta_4, \tag{2.34}$$

and under $\tau \to -\frac{1}{\tau}$:

$$\eta \longrightarrow (-i\tau)^{1/2}\eta, \tag{2.35}$$
$$\frac{\vartheta_1}{\eta} \longrightarrow e^{-i\pi/2}\frac{\vartheta_1}{\eta}, \tag{2.36}$$
$$\frac{\vartheta_2}{\eta} \longleftrightarrow \frac{\vartheta_4}{\eta}, \tag{2.37}$$
$$\frac{\vartheta_3}{\eta} \longrightarrow \frac{\vartheta_3}{\eta}. \tag{2.38}$$

Since the partition function is a product of the spin-structures of 64 fermions, the modular transformations will take us from one spin-structure to another. Modular invariance requires that both spin-structures related by these transformations be present in the partition function with equal weight. This gives the constraints

$$C\binom{\alpha}{\beta} = e^{i\frac{\pi}{4}(\alpha\cdot\alpha+\mathbf{1}\cdot\mathbf{1})}\, C\binom{\alpha}{\beta-\alpha+1}, \tag{2.39}$$
$$C\binom{\alpha}{\beta} = e^{i\frac{\pi}{4}\alpha\cdot\beta}\, C\binom{\beta}{\alpha}^{*}, \tag{2.40}$$

where $\mathbf{1}$ is the vector corresponding to periodic boundary conditions for all fermions and

$$\alpha\cdot\beta = \left\{\frac{1}{2}\sum_{\text{real left}} + \sum_{\text{complex left}} - \frac{1}{2}\sum_{\text{real right}} - \sum_{\text{complex right}}\right\}\alpha(f)\beta(f). \tag{2.41}$$

Another constraint arises when considering higher order loops :

$$C\binom{\alpha}{\beta}C\binom{\alpha'}{\beta'} = \delta_\alpha\delta_{\alpha'}e^{-i\frac{\pi}{2}\alpha\cdot\alpha'}C\binom{\alpha}{\beta+\alpha'}C\binom{\alpha'}{\beta'+\alpha}, \tag{2.42}$$

where δ_α is the space-time spin statistics index defined as

$$\delta_\alpha = \begin{cases} 1 \leftrightarrow \alpha(\psi^\mu_{1,2}) = 0, \\ -1 \leftrightarrow \alpha(\psi^\mu_{1,2}) = 1. \end{cases} \tag{2.43}$$

Now we can use these constraints to derive the rules for constructing a model.

Using (3.6) and (3.7) with $\alpha' = \alpha$ and $\beta = 0$ implies that

$$C\binom{\alpha}{0}^2 = \delta_\alpha C\binom{\alpha}{0}C\binom{0}{0}, \tag{2.44}$$

wich means that either $C\binom{\alpha}{0} = 0$ or $C\binom{\alpha}{0} = \delta_\alpha$, where we have normalized $C\binom{0}{0} = 1$. We define a set of vectors Ξ by

$$\Xi = \left\{\alpha \,\middle|\, C\binom{\alpha}{0} = \delta_\alpha\right\} \tag{2.45}$$

Using (3.6) and (3.7) we see that Ξ is an Abelian additive group, and spin-structures contributing to the partition function are pairs of elements in Ξ. Furthermore, if we take Ξ to be finite (and therefore the boundary conditions to be rationnal), it is in fact isomorphic to

$$\Xi \cong \bigoplus_{i=1}^{k} \mathbb{Z}_{N_i}, \tag{2.46}$$

which means that Ξ is generated by a set of basis vectors $\{b_1, \ldots, b_k\}$, such that

$$\sum_{i=1}^{k} m_i b_i = 0 \Leftrightarrow m_i = 0 \mod N_i \ \forall i, \tag{2.47}$$

where N_i is the smallest positive integer where $N_i b_i = 0$.

Now if we take three vectors α, β, $\gamma \in \Xi$, (3.7) can be rewritten as

$$C\begin{pmatrix}\alpha\\ \beta+\gamma\end{pmatrix} = \delta_\alpha C\begin{pmatrix}\alpha\\ \beta\end{pmatrix} C\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}. \tag{2.48}$$

Equation (3.5) with $\alpha = \beta$ gives

$$C\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} = e^{-i\frac{\pi}{4}\alpha\cdot\alpha} C\begin{pmatrix}\alpha\\ \mathbf{1}\end{pmatrix}. \tag{2.49}$$

Manipulating (3.6), (3.7), (3.8) and since β generates a finite group of order N_β, we show that if N_{ij} is the least common multiple of N_i and N_j, it must satisfy

$$N_{ij} b_i \cdot b_j = 0 \bmod 4. \tag{2.50}$$

When $i = j$ this constraint holds when N_i is odd. If it is even we have the stronger constraint :

$$N_i b_i^2 = 0 \bmod 8. \tag{2.51}$$

When all the constraints derived in this section are satisfied, the modular invariance condition is also satisfied, and thus there is no further obstruction to consistently assigning coefficients to pairs of elements of Ξ.

2.2.2 Deriving the spectrum of a model

We can rewrite equations (2.17) and (2.18) in the general case as

$$Z_F\begin{bmatrix}\alpha(f)\\ \beta(f)\end{bmatrix} = \mathrm{Tr}_\alpha \ \left[q^{H_\alpha} e^{\pi i \beta\cdot F_\alpha}\right], \tag{2.52}$$

where H_α is the Hamiltonian and F_α the fermion number operator in the Hilbert space sector $\mathscr{H}_\alpha$ defined by the vector α.

The partition function can then be written as a sum over sectors, using the fact that the basis vectors b_i are generators of a discrete group $\mathbb{Z}_{N_i}$ and applying equation (3.8) :

$$Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2} \; Z_B^2 \sum_{\alpha\in\Xi} \delta_\alpha \mathrm{Tr} \left\{ \prod_{b_i} \left(\delta_\alpha C\binom{\alpha}{b_i} e^{i\pi b_i\cdot F_\alpha} + \cdots \right.\right.$$

$$\left.\left. \cdots + \left\{ \delta_\alpha C\binom{\alpha}{b_i} e^{i\pi b_i\cdot F_\alpha} \right\}^{N_i-1} + 1 \right) e^{i\pi\tau H_\alpha} \right\}. \tag{2.53}$$

The only states that appear in the partition function are those that realise a generalised GSO projection

$$e^{i\pi b_i\cdot F_\alpha} \left|S\right\rangle_\alpha = \delta_\alpha C\binom{\alpha}{b_i}^* \left|S\right\rangle_\alpha . \tag{2.54}$$

The full Hilbert space is therefore

$$\mathscr{H} = \bigoplus_{\alpha\in\Xi} \prod_{i=1}^{k} \left\{ e^{i\pi b_i\cdot F_\alpha} = \delta_\alpha C\binom{\alpha}{b_i}^* \right\} .\mathscr{H}_\alpha \tag{2.55}$$

The mass of a state in the sector $\mathscr{H}_\alpha$ is given by the Virasoro gauge conditions

$$M_L^2 = -\frac{1}{2} + \frac{\alpha_L\cdot\alpha_L}{8} + \sum_{\text{left-movers}} \nu \tag{2.56}$$

$$M_R^2 = -1 + \frac{\alpha_R\cdot\alpha_R}{8} + \sum_{\text{right-movers}} \nu, \tag{2.57}$$

where $\alpha_{L,R}$ are the boundary conditions defined by the vector α for the left and right moving fermions and we should have $M_L^2 = M_R^2$ for each state. We can compute the frequencies of the fermionic oscillators depending on their boundary conditions :

$$f \to -e^{i\pi\alpha(f)} f \quad , \quad f^* \to -e^{-i\pi\alpha(f)} f^*. \tag{2.58}$$

The frequency for the fermions is then given by

$$\nu_f = \frac{1+\alpha}{2} \quad , \quad \nu_f^* = \frac{1-\alpha}{2}, \tag{2.59}$$

which can be rewritten using their fermion number operator F as

$$\nu = \frac{1+\alpha}{2} + F. \tag{2.60}$$

Each complex fermion f generates a $U(1)$ current, with a charge with respect to the unbroken Cartan generators of the four-dimensional gauge group given by

$$Q_\nu(f) = \nu - \frac{1}{2}, \tag{2.61}$$

$$= \frac{\alpha(f)}{2} + F. \tag{2.62}$$

2.3 Rules of construction of a model

Now that we have derived all the constraints to build a free-fermionic model we recall them in a set of rules called ABK rules, namely Antoniadis, Bachas and Kounnas who derived them in the two papers [1, 2]. These will be our working tools for the work described in the following parts, and an understanding of the previous derivations is not necessary to follow it.

2.3.1 The ABK rules

Rules on the basis vectors

The first thing we require is a set of basis vectors that defines Ξ, the space of all sectors. For each sector $\beta \in \Xi$ there is a Hilbert space of states. Each basis vector b_i consists of a set of boundary conditions for each fermion, written as

$$b_i = \left\{\alpha(\psi^\mu_{1,2}), \ldots, \alpha(w^6)|\alpha(y^1), \ldots, \alpha(\bar{\Phi}^8)\right\}, \tag{2.63}$$

where $\alpha(f)$ is defined by

$$f \to -e^{i\pi\alpha(f)} f. \tag{2.64}$$

The b_i have to form an additive group and satisfy the constraints derived in the previous section. If N_i is the smallest positive integer for which $N_i b_i = 0$ and N_{ij} is the least common multiple of N_i and N_j, the following rules must hold :

$$\begin{array}{ll} 1. & \sum_{i=1}^{k} m_i b_i = 0 \Leftrightarrow m_i = 0 \mod N_i \ \forall i, \\ 2. & \mathbf{1} \in \Xi, \\ 3. & N_{ij} b_i \cdot b_j = 0 \text{ mod } 4, \\ 4. & N_i b_i^2 = 0 \text{ mod } 8, \\ 5. & \text{even number of real fermions.} \end{array} \tag{2.65}$$

Rules on the coefficients

Once the space of states is defined we have to specify the phases $C\binom{b_i}{b_j}$ for all intersection of basis vectors. These coefficients are required to obey the constraints (3.6), (3.8) and (3.9), which can be rewritten as :

$$\begin{array}{ll} 1. & C\binom{b_i}{b_j} = \delta_{b_i}\ e^{\frac{2\pi i}{N_j} n} = \delta_{b_j}\ e^{\frac{\pi i}{2} b_i \cdot b_j}\ e^{\frac{2\pi i}{N_i} m}, \\ 2. & C\binom{b_i}{b_i} = e^{-i\frac{\pi}{4} b_i \cdot b_i}\ C\binom{b_i}{\mathbf{1}}, \\ 3. & C\binom{b_i}{b_j} = e^{i\frac{\pi}{2} b_i \cdot b_j}\ C\binom{b_j}{b_i}^*, \\ 4. & C\binom{b_i}{b_j + b_k} = \delta_{b_i}\ C\binom{b_i}{b_j}\ C\binom{b_i}{b_k}. \end{array} \tag{2.66}$$

GSO projections

There is only a limited number of physical states that realise a generalised GSO constraint. These states have to survive the GSO projection

$$e^{i\pi b_i \cdot F_\alpha} \left| S \right\rangle_\alpha = \delta_\alpha C \binom{\alpha}{b_i}^* \left| S \right\rangle_\alpha \quad \forall i, \tag{2.67}$$

where $\left| S \right\rangle_\alpha$ is a state in the sector $\alpha \in \Xi$.

Expliciting the massless spectrum

Now that these conditions are satisfied we can use the following formulae to analyse the complete spectrum.

The mass of a state in the sector α is given by

$$M_L^2 = -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + \sum_{left-movers} \nu, \tag{2.68}$$

$$M_R^2 = -1 + \frac{\alpha_R \cdot \alpha_R}{8} + \sum_{right-movers} \nu, \tag{2.69}$$

where $\alpha_{L,R}$ are the boundary conditions defined by the vector α for the left and right moving fermions and we should have $M_L^2 = M_R^2$.

The frequencies of the fermionic oscillators are

$$\nu_f = \frac{1+\alpha}{2} \quad , \quad \nu_f^* = \frac{1-\alpha}{2}, \tag{2.70}$$

and when a $U(1)$ gauge group is realised by a free fermion the charge under this $U(1)$ is

$$Q_\nu(f) = \frac{\alpha(f)}{2} + F. \tag{2.71}$$

2.3.2 A first example of a free fermionic model

Let us start to construct the simplest model to understand how this construction works.

Since the vector $\mathbf{1}$ (all conditions are periodic) is required to be in Ξ, we will start with the basis $\{\mathbf{1}\}$. We have two sectors : $\Xi = \{\mathbf{1}, 2 \cdot \mathbf{1} = \mathbf{0}\}$. We will use the notation NS for the sector $\mathbf{0}$, which is the Neveu-Schwartz sector.

$2 \cdot \mathbf{1} = \mathbf{0}$, so $N_{\mathbf{1}} = 2$ and $\mathbf{1} \cdot \mathbf{1} = -12$, therefore the rules on the basis vectors are satisfied.

Since the states with a mass at the string level would have a mass of the order of the Planck mass they are not phenomenologically acceptable. We will be especially concerned with the massless string spectrum, which must include all the particles of the Standard Model. We need to insure that there are no states with $M < 0$. Applying (2.68) we find in this simple model :

For the sector **1**

$$M_L^2 = -\frac{1}{2} + \frac{10}{8} + \sum_{left-movers} \nu > 0, \tag{2.72}$$

so this sector contains no massless states.

For the NS sector

$$M_L^2 = -\frac{1}{2} + \frac{0}{8} + \sum_{left-movers} \nu, \tag{2.73}$$

$$M_R^2 = -1 + \frac{0}{8} + \sum_{right-movers} \nu, \tag{2.74}$$

where for the fermions the frequency is given by

$$\nu_{f,f^*} = \frac{1 \pm 0}{2} = \frac{1}{2}. \tag{2.75}$$

We need to satisfy $M_L^2 = M_R^2$. Thus we either get a tachyonic state with a negative mass $-\frac{1}{2}$ by acting on the NS vacuum with one fermionic right-moving oscillator, or a massless state by acting with one left-moving fermionic oscillator and either two right-moving fermionic oscillators or one right-moving bosonic oscillator.

The massless states are then the following :

- $\psi^\mu_{1/2} \partial \bar{X}^\nu_1 \left|0\right\rangle_{NS}$, where $\partial \bar{X}^\nu_1$ is the bosonic creation operator. These states correspond to the graviton, the dilaton and the antisymmetric tensor.

- $\psi^\mu_{1/2} \bar{\Phi}^a_{1/2} \bar{\Phi}^b_{1/2} \left|0\right\rangle_{NS}\,,\ a, b \in \{1, \ldots, 44\}$: Gauge bosons in the adjoint representation of $SO(44)$.

- $\{\chi^i_{1/2}, y^i_{1/2}, w^i_{1/2}\} \partial \bar{X}^\nu_1 \left|0\right\rangle_{NS}\,,\ i \in \{1, \ldots, 6\}$: Gauge bosons in the adjoint representation of $SU(2)^6$.

- $\{\chi^i_{1/2}, y^i_{1/2}, w^i_{1/2}\} \bar{\Phi}^a_{1/2} \bar{\Phi}^b_{1/2} \left|0\right\rangle_{NS}\,,\ i \in \{1, \ldots, 6\}$: Scalars in the adjoint representation of $SU(2)^6 \times SO(44)$.

The tachyonic states are $\bar{\Phi}^a_{1/2} \left|0\right\rangle_{NS}$ with a mass $M^2 = -\frac{1}{2}$.

Now we have to perform the GSO projection for each state. For example for the first state we have

$$e^{i\pi \mathbf{0} \cdot F_\mathbf{1}} \psi^\mu_{1/2} \partial \bar{X}^\nu_1 \left|0\right\rangle_{NS} = -\psi^\mu_{1/2} \partial \bar{X}^\nu_1 \left|0\right\rangle_{NS}\,, \tag{2.76}$$

and using the rules (2.66) we compute $C\binom{NS}{\mathbf{1}} = \delta_{\mathbf{1}} = -1$, therefore

$$\delta_{NS} C\binom{NS}{\mathbf{1}}^* \psi^{\mu}_{1/2} \partial \bar{X}^{\nu}_1 \left|0\right\rangle_{NS} = -\psi^{\mu}_{1/2} \partial \bar{X}^{\nu}_1 \left|0\right\rangle_{NS} . \tag{2.77}$$

This state survives the GSO projection.

Similarly all other states in this model, including tachyons, survive the GSO projection. To get rid of the tachyons we need to add more basis vectors with appropriate phases, in order to project it out of the spectrum with the corresponding GSO projections. But we would also like to include the particule content of the Standard Model and reduce the gauge group. In the next part we will study models with these properties.

Part 3

Classification of flipped $\mathbf{SU(5)\times U(1)}$ heterotic string models

We now turn to a specific class of free-fermionic models. Models with a flipped $SU(5)\times U(1)$ gauge group have already been studied in [3, 4, 5]. Our approach is different since we use a general basis to generate a large class of models with the appropriate gauge group and study their content. We start by explaining the choice of basis vectors before studying the gauge bosons in the spectrum of these models. We derive the conditions for possible enhancements of this gauge group. Then we compute analytical expressions for the generalised GSO projections of all sectors, that allow to study the possible matter content in the spectrum of all these models.

3.1 Construction of flipped $\mathbf{SU(5)\times U(1)}$ models

3.1.1 A general basis for models with an $\mathbf{SO(10)}$ gauge group

As explained in the previous part, we have to specify a set of basis vectors to build a free-fermionic model. A class of free-fermionic constructions corresponds to $Z_2 \times Z_2$ toroidal orbifolds compactification. The free-fermionic $Z_2 \times Z_2$ orbifolds preserve the SO(10) GUT embedding of the Standard Model spectrum and a classification of these models has been undertaken in [11, 12]. In [12] a single basis is used to generate the different classes of models. Our work will be based on the same basis, which we explicit here.

We use the notations introduced in Part 2 for the world-sheet fermions :

$$\{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6}|\bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\Phi}^{1,\dots,8}\}. \tag{3.1}$$

We start with the basis :

$$\begin{aligned} v_1 = \mathbf{1} &= \{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6}|\bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\Phi}^{1,\dots,8}\}, \\ v_2 = S &= \{\psi^\mu, \chi^{1,\dots,6}\}, \\ v_{2+i} = e_i &= \{y^i, \omega^i|\bar{y}^i, \bar{\omega}^i\},\ i = 1, \dots, 6, \\ v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56}|\bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^1, \bar{\psi}^{1,\dots,5}\}, \\ v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56}|\bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^2, \bar{\psi}^{1,\dots,5}\}, \\ v_{11} = z_1 &= \{\bar{\Phi}^{1,\dots,4}\}, \\ v_{12} = z_2 &= \{\bar{\Phi}^{5,\dots,8}\}, \end{aligned} \tag{3.2}$$

where the fermions between brakets are those with R-boundary conditions (periodic) and those not quoted have NS-boundary conditions (antiperiodic).

The vectors $\mathbf{1}, S$ generate an $\mathcal{N} = 4$ supersymmetric model, with $SO(44)$ gauge symmetry. The vectors e_i, $i = 1, \dots, 6$ give rise to all possible symmetric shifts of the six internal fermionized coordinates; they break the SO(44) gauge group but preserve $\mathcal{N} = 4$ supersymmetry. The vectors b_1 and b_2 define the $SO(10)$ gauge symmetry and the $Z_2 \times Z_2$ orbifold twists, which break $\mathcal{N} = 4$ to $\mathcal{N} = 1$ supersymmetry. The remaining fermions not affected by the action of the previous vectors are $\bar{\Phi}^i$, $i = 1, \dots, 8$, which normally give rise to the hidden gauge group. The vectors z_1, z_2 reduce the untwisted gauge group from $SO(16)$ to $SO(8) \times SO(8)$. This choice of basis is the most general set of basis vectors, with symmetric shifts for the internal fermions, compatible with a suitable $SO(10)$ embedding.

Once we have performed GGSO projections for massless states on models constructed with this basis we end up with gauge bosons belonging to the adjoint representation of $SO(10) \times U(1)^3 \times SO(8)^2$, where we can identify $SO(10)$ with the gauge group under which observable matter is charged, because SO(10) contains $SU(3) \times SU(2) \times U(1)$ as a subgroup, and the $U(1)^3 \times SO(8)^2$ is supposed to convey interactions for hidden matter.

3.1.2 A general basis for flipped $\mathbf{SU(5)} \times \mathbf{U(1)}$ models

Now we are interested in flipped $SU(5) \times U(1)$ models so we have to break the gauge group from $SO(10)$ to $SU(5) \times U(1)$. This is done by adding a vector α to the previous basis. There are several possible choices for α that satisfy the rules on basis vectors, but we have to add a condition : it should be possible to project all enhancements of the gauge group from new sectors out in order to keep only flipped $SU(5) \times U(1)$ models. We take

$$\alpha = \{\bar{\psi}^{1,\dots,5} = \frac{1}{2}, \bar{\eta}^{1,2,3} = \frac{1}{2}, \bar{\Phi}^{1,\dots,4} = \frac{1}{2}, \bar{\Phi}^5 = 1\}, \tag{3.3}$$

where the condition $\bar{\psi}^{1,\dots,5} = \frac{1}{2}$ breaks $SO(10)$ to $SU(5) \times U(1)$ and the other conditions are required in order for the rules on basis vectors to be satisfied.

This time we have $4 \cdot \alpha = 0$, but $2 \cdot \alpha \neq 0$ because of the $\frac{1}{2}$ boundary conditions, so we also have to consider the sector 2α. We observe that

$$x \equiv 2\alpha + z_1 = 1 + S + \sum_{i=1}^{6} e_i + z1 + z2, \tag{3.4}$$

so that $\{v_{1\ldots12}, \alpha\}$ is not linear independent. Therefore we have to remove one of the vectors from the basis. To keep a symmetry we decide to remove the vector $v_1 = \mathbf{1}$. Our basis is now $\{v_{2\ldots12}, \alpha\}$.

3.1.3 Generalised GSO projections

The second ingredient that is needed to define the string vacuum are the generalised GSO projection coefficients that appear in the one-loop partition function, $c\binom{v_i}{v_j}$, spanning a 12×12 matrix. We only have to choose the elements with $i \geq j$, while the others are fixed by modular invariance. A priori there are therefore 78 independent coefficients corresponding to different string vacua. They can take the values ±1 when the two vectors have periodic or antiperiodic boundary conditions, and ±1 or $\pm\, i$ when α is one of the two vectors. Coefficients on the diagonal are also related by modular invariance, which gives rise to eleven constraints :

$$C\binom{\alpha}{\alpha} = -\prod_{i=1}^{6} C\binom{\alpha}{e_i} C\binom{\alpha}{z_2}, \tag{3.5}$$

$$C\binom{b_k}{b_k} = -\prod_{i=1}^{6} C\binom{b_k}{e_i} C\binom{b_k}{z_2}, \quad k = 1, 2, \tag{3.6}$$

$$C\binom{z_1}{z_1} = -\prod_{i=1}^{6} C\binom{z_1}{e_i} C\binom{z_1}{z_2}, \tag{3.7}$$

$$1 = \prod_{i=1}^{6} C\binom{S}{e_i} C\binom{S}{z_2}, \tag{3.8}$$

$$1 = \prod_{\substack{i=1 \\ i\neq k}}^{6} C\binom{e_k}{e_i} C\binom{e_k}{z_2}, \quad k = 1\ldots6. \tag{3.9}$$

Eleven coefficients are fixed by requiring that the models possess $\mathcal{N} = 1$ supersymmetry. Without loss of generality we impose the associated GGSO projection coefficients

$$C\binom{S}{e_i} = C\binom{S}{b_m} = C\binom{S}{z_n} = C\binom{S}{\alpha} = -1, \tag{3.10}$$
$$i = 1, ..., 6,\ m = 1, 2,\ n = 1, 2.$$

We remark here that there may exist some degeneracies in this space of physical vacua with respect to the properties of the effective low energy field theory.

3.2 Enhancements of the gauge group

Now that the basis is defined and that each possible choice of GSO phases corresponds to a model, let us have a look at the gauge group. The vector bosons from the Neveu-Schwartz sector generate an

$$SU(5) \times U(1) \times U(1)^3 \times SU(4) \times U(1) \times U(1) \times SO(6) \tag{3.11}$$

gauge symmetry. Depending on the choices of the projection coefficients, extra gauge bosons may arise from the following sectors :

$$\mathbf{G} = \left\{ \begin{array}{ccccc} z_1, & z_2, & z_1 + z_2, & 2\alpha + z_1, & \alpha, \\ \alpha + z_1, & \alpha + z_2, & \alpha + z_1 + z_2, & -\alpha, & -\alpha + z_1, \\ -\alpha + z_2, & \alpha + z_1 + z_2, & -\alpha + z_1 + z_2 & & \end{array} \right\}. \tag{3.12}$$

Vector bosons that arise from these sectors may transform under a subgroup of the Neveu-Schwartz gauge group and enhance the untwisted gauge symmetry. Some of these enhancements might lead us back to an $SO(10)$ (or other groups) observable gauge group, which is not what we intend to study, or only enhance the hidden gauge group. We impose the condition that the only space-time vector bosons that remain in the spectrum are those that arise from the untwisted sector.

We now present the type of enhancements that can occur from different sectors with the conditions on the GGSO coefficients that allow them. Enhancements with conditions that do not satisfy the constraints (3.5)-(3.9) are not possible and are not presented here. We introduce a new notation for the GGSO coefficients :

$$C\binom{b_i}{b_j} = e^{i\pi(b_i|b_j)}, \quad (b_i|b_j) \in \{0, \frac{1}{2}, -\frac{1}{2}, 1\}. \tag{3.13}$$

Sector $2\alpha + z_1$

Enhancement Condition	Resulting Enhancement
$(2\alpha + z_1\vert e_i) = (2\alpha + z_1\vert z_k) = (2\alpha + z_1\vert\alpha) =$ $(2\alpha + z_1\vert b_k) = 0$	$SU(5) \times U(1) \times U(1)^3$ $\rightarrow SU(6) \times SU(2) \times U(1)^2$
$(2\alpha + z_1\vert e_i) = (2\alpha + z_1\vert z_k) = (2\alpha + z_1\vert\alpha) = 0$ $((2\alpha + z_1\vert b_1), (2\alpha + z_1\vert b_2)) \neq (0, 0)$	$SU(5) \times U(1) \times U(1)^3$ $\rightarrow SO(10) \times U(1)^3$
$(2\alpha + z_1\vert e_i) = (2\alpha + z_1\vert z_k) = (2\alpha + z_1\vert b_k) = 0$ $(2\alpha + z_1\vert\alpha) = 1$	$SU(5) \times U(1) \times U(1)^3$ $\rightarrow SO(10) \times U(1)^3$
$(2\alpha + z_1\vert e_i) = (2\alpha + z_1\vert z_k) = (2\alpha + z_1\vert\alpha) = 0$ $(2\alpha + z_1\vert\alpha) = 1$ $((2\alpha + z_1\vert b_1), (2\alpha + z_1\vert b_2)) \neq (0, 0)$	$SU(5) \times U(1) \times U(1)^3$ $\rightarrow SU(6) \times SU(2) \times U(1)^2$

The relations above hold for all $i = 1 \ldots 6$ and $k = 1, 2$. Note that the general condition to have an enhancement in this sector can be summarized as $(z_1|e_i) = 0$ and $(z_2|e_1) = 1$.

Sector $z_1 + z_2$

Enhancement Condition	Resulting Enhancement
$(z_1+z_2\|e_i) = (z_1+z_2\|b_k) = 0$	$SU(4)\times U(1)\times SO(6)\times U(1)$ $\rightarrow SU(8)\times U(1)$

The relations above hold for all $i = 1\dots6$ and $k = 1,2$.

Sector α

According to the formula (2.68) and using the expression of the frequency of fermionic oscillators (2.70) we observe that we need to add either one oscillator of frequency $\frac{1}{2}$ or two oscillators with frequency $\frac{1}{4}$ acting on the vacuum in order to obtain massless states.

States with two oscillators $\frac{1}{4}$:

Enhancement Condition	Resulting Enhancement
$(\alpha\|e_i) = (\alpha\|z_1) = (\alpha\|b_k) = 0$	$SU(4)\times SU(5)\times U(1)^3\times U(1)$ $\rightarrow SU(9)\times SU(2)\times U(1)^2$
$(\alpha\|e_i) = (\alpha\|z_1) = 0$ $((\alpha\|b_1),(\alpha\|b_2)) \neq (0,0)$	$SU(5)\times U(1)\times U(1)^3\times U(1)$ $\rightarrow SU(5)\times SU(3)\times U(1)^3$
$(\alpha\|e_i) = 0$ $(\alpha\|z_1) = 1$ $((\alpha\|b_1),(\alpha\|b_2)) \neq (1,1)$	$SU(4)\times U(1)\times U(1)\times U(1)$ $\rightarrow SU(4)\times SU(2)\times SU(2)\times U(1)$
$(\alpha\|e_i) = 0$ $(\alpha\|z_1) = (\alpha\|b_k) = 1$	$SU(5)\times U(1)\times SU(4)\times U(1)\times U(1)$ $\rightarrow SU(7)\times SU(3)\times SU(2)\times SU(2)$

The relations above hold for all $i = 1\dots6$ and $k = 1,2$.

States with one oscillator $\frac{1}{2}$:

Enhancement Condition	Resulting Enhancement
$(\alpha\|e_i) = (\alpha\|z_1) = (\alpha\|b_k) = 0$	$SO(6)\times U(1)\rightarrow SO(7)\times U(1)$
$(\alpha\|e_j) = (\alpha\|z_1) = 0$ $(\alpha\|e_i) = 1$ AND $(\alpha\|b_1) = 0,\ i = 1,2$ or $(\alpha\|b_2) = 0,\ i = 3,4$ or $(\alpha\|b_1) = (\alpha\|b_2),\ i = 5,6$	$U(1)\rightarrow U(1)$

The relations above hold for all $i, j = 1\dots6$ where $i \neq j$ and $k = 1,2$.

Sector $\alpha + z_1$

States with two oscillators $\frac{1}{4}$:

Enhancement Condition	Resulting Enhancement
$(\alpha+z_1\|e_i)=(\alpha+z_1\|z_1)=(\alpha+z_1\|b_k)=0$ $(\alpha+z_1\|z_2)=(\alpha+z_1\|\alpha)+1$	$SU(4)\times SU(5)\times U(1)^3\times U(1)$ $\to SU(9)\times SU(2)\times U(1)^2$
$(\alpha+z_1\|e_i)=(\alpha+z_1\|z_1)=0$ $((\alpha+z_1\|b_1),(\alpha+z_1\|b_2))\neq(0,0)$ $(\alpha+z_1\|z_2)=(\alpha+z_1\|\alpha)+1$	$SU(5)\times U(1)\times U(1)^3\times U(1)$ $\to SU(5)\times SU(3)\times U(1)^3$
$(\alpha+z_1\|e_i)=0$ $(\alpha+z_1\|z_1)=1$ $((\alpha+z_1\|b_1),(\alpha+z_1\|b_2))\neq(1,1)$ $(\alpha+z_1\|z_2)=(\alpha+z_1\|\alpha)$	$SU(4)\times U(1)\times U(1)\times U(1)$ $\to SU(4)\times SU(2)\times SU(2)\times U(1)$
$(\alpha+z_1\|e_i)=0$ $(\alpha+z_1\|z_1)=(\alpha+z_1\|b_k)=1$ $(\alpha+z_1\|z_2)=(\alpha+z_1\|\alpha)$	$SU(5)\times U(1)\times SU(4)\times U(1)\times U(1)$ $\to SU(7)\times SU(3)\times SU(2)\times SU(2)$

The relations above hold for all $i=1\dots6$ and $k=1,2$.

States with one oscillator $\frac{1}{2}$:

Enhancement Condition	Resulting Enhancement
$(\alpha+z_1\|e_i)=(\alpha+z_1\|z_1)=(\alpha+z_1\|b_k)=0$ $(\alpha+z_1\|z_2)=(\alpha+z_1\|\alpha)+1$	$SO(6)\times U(1)\to SO(7)\times U(1)$
$(\alpha+z_1\|e_j)=(\alpha+z_1\|z_1)=0$ $(\alpha+z_1\|e_i)=1$ $(\alpha+z_1\|z_2)=(\alpha+z_1\|\alpha)$ AND $(\alpha+z_1\|b_1)=0,\ i=1,2$ or $(\alpha+z_1\|b_2)=0,\ i=3,4$ or $(\alpha+z_1\|b_1)=(\alpha+z_1\|b_2),\ i=5,6$	$U(1)\to U(1)$

The relations above hold for all $i,j=1\dots6$ where $i\neq j$ and $k=1,2$.

Sectors $-\alpha,\ -\alpha+z_1$

These sectors give the same conditions and the same enhancements as the sectors α, $\alpha+z_1$.

Sector $\alpha + z_2$

Enhancement Condition	Resulting Enhancement
$(\alpha + z_2\|e_i) = (\alpha + z_2\|z_1) = 0$ $(\alpha + z_2\|\alpha) = \frac{1}{2}$ $(\alpha + z_2\|b_k) = 1$	$SO(6) \times SU(5) \times U(1)$ $\to SO(11) \times SU(2) \times U(1)^2$
$(\alpha + z_2\|e_i) = (\alpha + z_2\|z_1) = 0$ $(\alpha + z_2\|\alpha) = \frac{1}{2}$ $((\alpha + z_2\|b_1), (\alpha + z_2\|b_2)) \neq (1, 1)$	$SO(6) \times U(1) \to SO(5) \times SU(5)$
$(\alpha + z_2\|e_i) = (\alpha + z_2\|b_k) = 0$ $(\alpha + z_2\|\alpha) = \frac{1}{2}$ $(\alpha + z_2\|z_1) = 1$	$SO(6) \times SO(4) \times U(1) \to SO(10) \times U(1)^2$

The relations above hold for all $i = 1 \ldots 6$ and $k = 1, 2$.

Sector $-\alpha + z_2$

Enhancement Condition	Resulting Enhancement
$(-\alpha + z_2\|e_i) = (-\alpha + z_2\|z_1) = 0$ $(-\alpha + z_2\|\alpha) = -\frac{1}{2}$ $(\alpha + z_2\|b_k) = 1$	$SO(6) \times SU(5) \times U(1)$ $\to SO(11) \times SU(2) \times U(1)^2$
$(-\alpha + z_2\|e_i) = (-\alpha + z_2\|z_1) = 0$ $(-\alpha + z_2\|\alpha) = -\frac{1}{2}$ $((-\alpha + z_2\|b_1), (-\alpha + z_2\|b_2)) \neq (1, 1)$	$SO(6) \times U(1) \to SO(5) \times SU(5)$
$(-\alpha + z_2\|e_i) = (-\alpha + z_2\|b_k) = 0$ $(-\alpha + z_2\|\alpha) = -\frac{1}{2}$ $(-\alpha + z_2\|z_1) = 1$	$SO(6) \times SO(4) \times U(1) \to SO(10) \times U(1)^2$

The relations above hold for all $i = 1 \ldots 6$ and $k = 1, 2$.

Sector $\alpha + z_1 + z_2$

Enhancement Condition	Resulting Enhancement
$(\alpha + z_1 + z_2\|e_i) = (\alpha + z_1 + z_2\|z_1) = 0$ $(\alpha + z_1 + z_2\|\alpha) = \frac{1}{2}$ $(\alpha + z_1 + z_2\|b_k) = 1$	$SO(6) \times SU(5) \times U(1) \to SO(11) \times SU(2) \times U(1)^2$
$(\alpha + z_1 + z_2\|e_i) = (\alpha + z_1 + z_2\|z_1) = 0$ $(\alpha + z_1 + z_2\|\alpha) = \frac{1}{2}$ $((\alpha + z_1 + z_2\|b_1), (\alpha + z_1 + z_2\|b_2)) \neq (1, 1)$	$SO(6) \times U(1) \to SO(5) \times SU(5)$
$(\alpha + z_1 + z_2\|e_i) = (\alpha + z_1 + z_2\|b_k) = 0$ $(\alpha + z_1 + z_2\|\alpha) = -\frac{1}{2}$ $(\alpha + z_1 + z_2\|z_1) = 1$	$SO(6) \times SO(4) \times U(1) \to SO(10) \times U(1)^2$

The relations above hold for all $i = 1 \ldots 6$ and $k = 1, 2$.

Sector $-\alpha+z_1+z_2$

Enhancement Condition	Resulting Enhancement
$(-\alpha+z_1+z_2\|e_i)=(-\alpha+z_1+z_2\|z_1)=0$ $(-\alpha+z_1+z_2\|\alpha)=-\frac{1}{2}$ $(-\alpha+z_1+z_2\|b_k)=1$	$SO(6)\times SU(5)\times U(1)$ $\rightarrow SO(11)\times SU(2)\times U(1)^2$
$(-\alpha+z_1+z_2\|e_i)=(-\alpha+z_1+z_2\|z_1)=0$ $(-\alpha+z_1+z_2\|\alpha)=-\frac{1}{2}$ $((-\alpha+z_1+z_2\|b_1),(-\alpha+z_1+z_2\|b_2))\neq(1,1)$	$SO(6)\times U(1)\rightarrow SO(5)\times SU(5)$
$(-\alpha+z_1+z_2\|e_i)=(-\alpha+z_1+z_2\|b_k)=0$ $(-\alpha+z_1+z_2\|\alpha)=\frac{1}{2}$ $(-\alpha+z_1+z_2\|z_1)=1$	$SO(6)\times SO(4)\times U(1)\rightarrow SO(10)\times U(1)^2$

The relations above hold for all $i=1\ldots6$ and $k=1,2$.

Sector z_1

Enhancement Condition	Resulting Enhancement
$(z_1\|e_i)=(z_1\|b_k)=(z_1\|\alpha)=0$ $(z_1\|z_2)=1$	$SO(6)\times U(1)\times SU(4)\times U(1)$ $\rightarrow SO(8)\times SO(8)$
$(z_1\|e_i)=(z_1\|b_k)=0$ $(z_1\|z_2)=(z_1\|\alpha)=1$	$SO(6)\times U(1)\times SU(4)\times U(1)$ $\rightarrow SO(12)\times SU(2)\times SU(2)$
$(z_1\|e_i)=(z_1\|z_2)=0$ $((z_1\|b_1),(z_1\|b_2))\neq(1,1)$	$SU(4)\times U(1)\times U(1)$ $\rightarrow SU(5)\times U(1)$
$(z_1\|e_i)=(z_1\|z_2)=0$ $(z_1\|b_1)=(z_1\|b_2)=1$	$SU(4)\times U(1)\times SU(5)\times U(1)$ $\rightarrow SU(9)\times U(1)$
$(z_1\|e_j)=(z_1\|z_2)=(z_1\|\alpha)=0$ $(z_1\|e_i)=1$ AND $(z_1\|b_1)=0,\ i=1,2$ or $(z_1\|b_2)=0,\ i=3,4$ or $(z_1\|b_1)=(z_1\|b_2),\ i=5,6$	$U(1)\rightarrow SU(2)$
$(z_1\|e_j)=(z_1\|z_2)=0$ $(z_1\|e_i)=(z_1\|\alpha)=1$ AND $(z_1\|b_1)=0,\ i=1,2$ or $(z_1\|b_2)=0,\ i=3,4$ or $(z_1\|b_1)=(z_1\|b_2),\ i=5,6$	$SU(4)\rightarrow S0(7)$

The relations above, hold for all $i,j=1\ldots6$ where $i\neq j$ and $k=1,2$.

Sector z_2

Enhancement Condition	Resulting Enhancement
$(z_2\vert e_i)=(z_2\vert b_k)=0$ $(z_2\vert z_1)=1$	$SO(6)\times U(1)\times SU(4)\times U(1)$ $\to SU(8)\times U(1)$
$(z_2\vert e_i)=(z_2\vert z_1)=0$ $((z_2\vert b_1),(z_2\vert b_2))\neq(1,1)$	$SO(6)\times U(1)\times U(1)$ $\to SU(5)\times U(1)$
$(z_2\vert e_i)=(z_2\vert z_1)=0$ $(z_2\vert b_1)=(z_2\vert b_2)=1$	$SO(6)\times U(1)\times SU(5)\times U(1)$ $\to SU(9)\times U(1)$

The relations above hold for all $i=1\dots6$ and $k=1,2$.

We now impose in this part that the set of GGSO coefficients for studied models does not satisfy any of these conditions for enhancements.

3.3 Matter content and expression of the projectors

We analyse the matter content in the different sectors. When we consider different models (different choices of GGSO coefficient) the matter content varies, since some states are projected out by the GGSO projections. The conditions for a sector to provide states in a sector can be expressed in term of projectors P(GGSO coefficients), where $P=0$ means that the sector does not provides states and $P=1$ means that the state is present in the spectrum.

3.3.1 Chiral matter

The chiral matter spectrum arises from the twisted sectors. The chiral spinorial representations of the observable $SU(5)\times U(1)$ arise from the sectors :

$$\begin{aligned}
B^{(1)}_{pqrs} &= S+b_1+pe_3+qe_4+re_5+se_6,\\
&= \{\psi^\mu,\chi^{12},(1-p)y^3\bar{y}^3,p\omega^3\bar{\omega}^3,(1-q)y^4\bar{y}^4,q\omega^4\bar{\omega}^4,\\
&\qquad (1-r)y^5\bar{y}^5,r\omega^5\bar{\omega}^5,(1-s)y^6\bar{y}^6,s\omega^6\bar{\omega}^6,\bar{\eta}^1,\bar{\psi}^{1..5}\},\\
B^{(2)}_{pqrs} &= S+b_2+pe_1+qe_2+re_5+se_6,\\
B^{(3)}_{pqrs} &= S+b_3+pe_1+qe_2+re_3+se_4,
\end{aligned}\tag{3.14}$$

where $p,q,r,s=0,1$ and $b_3=b_1+b_2+2\alpha+z_1$. These 48 sectors give rise to $\mathbf{16}$ and $\overline{\mathbf{16}}$ representations of $SO(10)$ decomposed under $SU(5)\times U(1)$:

$$\mathbf{16}=(\bar{\mathbf{5}},-\frac{3}{2})+(\mathbf{10},\frac{1}{2})+(\mathbf{1},\frac{5}{2}),\tag{3.15}$$

$$\overline{\mathbf{16}}=(\mathbf{5},\frac{3}{2})+(\overline{\mathbf{10}},-\frac{1}{2})+(\mathbf{1},-\frac{5}{2}).\tag{3.16}$$

The charges of the different states under the Cartan generators associated with complex world-sheet fermions can be computed using Equation (2.71). The states here have charges Q_1, Q_2, Q_3, Q_4, Q_5 equal to $\pm\frac{1}{2}$ under the observable gauge group, so they should be linked to hypercharge and electromagnetic charge. We used the following normalisations for the hypercharge and the electromagnetic charge :

$$Y = -\frac{1}{5}U(1)_5 + \frac{2}{5}\tilde{Y}, \tag{3.17}$$

$$Q_{em} = Y + T_3, \tag{3.18}$$

where $\tilde{Y}$ is the trace of the charges and T_3 the isospin of the weak symmetry :

$$U(1)_5 = \frac{1}{3}(Q_1 + Q_2 + Q_3) - \frac{1}{2}(Q_4 + Q_5), \tag{3.19}$$

$$\tilde{Y} = Q_1 + Q_2 + Q_3 + Q_4 + Q_5, \tag{3.20}$$

$$T_3 = \frac{1}{2}(Q_4 - Q_5), \tag{3.21}$$

which gives

$$Y = \frac{1}{3}(Q_1 + Q_2 + Q_3) + \frac{1}{2}(Q_4 + Q_5), \tag{3.22}$$

$$Q_{em} = Y + \frac{1}{2}(Q_4 - Q_5). \tag{3.23}$$

The following table summarises the eigenvalues of the electroweak $SU(2) \times U(1)$ Cartan generators, in respect to states which fall into the chiral observable $SU(5) \times U(1)$ representations :

Representation	$\bar{\psi}^{1,2,3}$	$\bar{\psi}^{4,5}$	Y	Q_{em}
$(\mathbf{5}, \frac{3}{2})$	$(+,+,+)$	$(+,-)$	1/2	1,0
	$(+,+,-)$	$(+,+)$	2/3	2/3
$(\bar{\mathbf{5}}, -\frac{3}{2})$	$(+,-,-)$	$(-,-)$	-2/3	-2/3
	$(-,-,-)$	$(+,-)$	-1/2	-1,0
$(\mathbf{10}, \frac{1}{2})$	$(+,+,+)$	$(-,-)$	0	0
	$(+,-,-)$	$(+,+)$	1/3	1/3
	$(+,+,-)$	$(+,-)$	1/6	-1/3,2/3
$(\overline{\mathbf{10}}, -\frac{1}{2})$	$(+,+,-)$	$(-,-)$	-1/3	-1/3
	$(+,-,-)$	$(+,-)$	-1/6	1/3,-2/3
	$(-,-,-)$	$(+,+)$	0	0
$(\mathbf{1}, \frac{5}{2})$	$(+,+,+)$	$(+,+)$	1	1
$(\mathbf{1}, -\frac{5}{2})$	$(-,-,-)$	$(-,-)$	-1	-1

In the previous table, $''+''$ and $''-''$ label the contribution of an oscillator with fermion number $F = 0$ or $F = -1$ to the degenerate vacuum. The case of $(+,-,-)$ under $\bar{\psi}^{1,2,3}$ for example, corresponds to a part of the Ramond vacuum formed by one oscillator with fermion number $F = 0$ and two oscillators with fermion numbers $F = -1$.

These states correspond to particles of the Standard Model. More precisely we can decompose these representations under $SU(3) \times SU(2) \times U(1)$:

$$(\bar{\mathbf{5}}, -\frac{3}{2}) = (\bar{3}, 1, -\frac{2}{3})_{u^c} + (1, 2, -\frac{1}{2})_L, \tag{3.24}$$

$$(\mathbf{10}, \frac{1}{2}) = (3, 2, \frac{1}{6})_Q + (\bar{3}, 1, \frac{1}{3})_{d^c} + (1, 1, 0)_{\nu^c}, \tag{3.25}$$

$$(\mathbf{1}, \frac{5}{2}) = (1, 1, 1)_{e^c}, \tag{3.26}$$

where L is the lepton doublet and Q the quark doublet.

A phenomenologically viable model must consist of 3 families, so we have to count the number of $\mathbf{16}$ and $\overline{\mathbf{16}}$. The choice of GGSO coefficient will determine the model we consider and the number of families. In order to be able to distinguish between $\mathbf{16}$ and $\overline{\mathbf{16}}$, one has to define operators that will determine the representations in which the states of each observable sector will fall into. The operators $X^{(i)_{SU(5)}}_{pqrs} = \pm 1$ that define the $SU(5)$ chirality ($\mathbf{5}$ or $\bar{\mathbf{5}}$) for B^1_{pqrs}, B^2_{pqrs} and B^3_{pqrs} are given by combining the constraints arising on the chirality $Ch(\bar{\psi}^{1..5})$ when applying the GGSO projections :

$$\begin{aligned} X^{(1)_{SU(5)}}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{b_2 + (1-r)e_5 + (1-s)e_6}, \\ X^{(2)_{SU(5)}}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{b_1 + (1-r)e_5 + (1-s)e_6}, \\ X^{(3)_{SU(5)}}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{b_1 + (1-r)e_3 + (1-s)e_4}. \end{aligned} \tag{3.27}$$

The states in the sector $B^{(1)}_{pqrs}$ can be projected out of the spectrum by the GGSO projection of the vectors b_1, b_2, z_1 and z_2. Similarly for all sectors, we can define a projector P such that the states survive when $P = 1$:

$$P^{(1)}_{pqrs} = \frac{1}{16}\left(1 - C\binom{e_1}{B^{(1)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_2}{B^{(1)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_1}{B^{(1)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(1)}_{pqrs}}\right) \tag{3.28}$$

$$P^{(2)}_{pqrs} = \frac{1}{16}\left(1 - C\binom{e_3}{B^{(2)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_4}{B^{(2)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_1}{B^{(2)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(2)}_{pqrs}}\right) \tag{3.29}$$

$$P^{(3)}_{pqrs} = \frac{1}{16}\left(1 - C\binom{e_5}{B^{(3)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_6}{B^{(3)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_1}{B^{(3)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(3)}_{pqrs}}\right) \tag{3.30}$$

These projectors can be expressed as a system of linear equations with p, q, r and s as unknowns. The solutions of a specific system of equations yield the different combinations of p, q, r and s for which sectors survive the GSO projections. We use the notation defined in (3.13) for the coefficients. The analytic expressions for each different projector $P^{1,2,3}_{pqrs}$ respectively, are given in a matrix form $\Delta^i W^i = Y^i$.

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1) \\ (e_2|b_1) \\ (z_1|b_1) \\ (z_2|b_1) \end{pmatrix} \tag{3.31}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2) \\ (e_4|b_2) \\ (z_1|b_2) \\ (z_2|b_2) \end{pmatrix} \tag{3.32}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3) \\ (e_6|b_3) \\ (z_1|b_3) \\ (z_2|b_3) \end{pmatrix} \tag{3.33}$$

3.3.2 Spinorial representations

States also arise from different sectors, with a different matter content. If we want a model to reproduce exactly the content of the Standard Model we should project them out of the spectrum. We can also keep hidden matter, but massless states with fractional electric charge should be eliminated. Indeed there are restrictive experimental constraints on these exotic states : the lightest fractionally charged state is necessarily stable and overproduced in the evolution of the early universe. Consequently it must be sufficiently massive and diluted to avoid constraints from contemporary searches.

We start by looking at spinorial representations. We have states corresponding to the representation **4** of $SO(6)$ in the sectors

$$B^{(4+,5+,6+)}_{pqrs} = B^{(1,2,3)}_{pqrs} + \alpha + z_2 \tag{3.34}$$
$$B^{(4-,5-,6-)}_{pqrs} = B^{(1,2,3)}_{pqrs} - \alpha + z_2. \tag{3.35}$$

And similar states in the sectors

$$B^{(7+,8+,9+)}_{pqrs} = B^{(1,2,3)}_{pqrs} + \alpha + z_1 + z_2, \tag{3.36}$$
$$B^{(7-,8-,9-)}_{pqrs} = B^{(1,2,3)}_{pqrs} - \alpha + z_1 + z_2. \tag{3.37}$$

States corresponding to the representation **8** of $SU(4) \times U(1)$ arise from the sectors

$$B^{(10,11,12)}_{pqrs} = B^{(1,2,3)}_{pqrs} + 2\alpha. \tag{3.38}$$

Finally states corresponding to the representation **8** of $SU(6) \times U(1)$ arise from the sectors

$$B^{(13,14,15)}_{pqrs} = B^{(1,2,3)}_{pqrs} + 2\alpha + z_1 + z_2. \tag{3.39}$$

The projectors for all these states, as well as their matrix formulation, are given in Appendix A.1.

3.3.3 Vectorial representations

Massless states are obtained in some sectors by acting on the vacuum with a Neveu-Schwarz right-moving fermionic oscillator. Vectorial representations can arise from the sectors

$$B_{pqrs}^{(16+,17+,18+)} = B_{pqrs}^{(1,2,3)} + \alpha, \tag{3.40}$$

and produce the following representations :

- $\{\bar{\eta}^1\}|R >_{pqrs}^{(i)}$, $i = 1, 2, 3$, where $|R >_{pqrs}^{(i)}$ is the degenerated Ramond vacuum of the $B_{pqrs}^{(i)}$ sector. These states transform as a vectorial representation of U(1).
- $\{\bar{\eta}^{2*}\}|R >_{pqrs}^{(i)}$ $and \{\bar{\eta}^{3*}\}|R >_{pqrs}^{(i)}$, $i = 1, 2, 3$. These states transform as a vectorial representation of U(1).
- $\{\bar{\psi}^{1...5}\}|R >_{pqrs}^{(i)}$, $i = 1, 2, 3$. These states transform as a vectorial representation of $SU(5) \times U(1)$.
- $\{\bar{\Phi}^{1..4*}\}|R >_{pqrs}^{(i)}$, $i = 1, 2, 3$. These states transform as a vectorial representation of $SU(4) \times U(1)$.

The same states with conjugate oscillators are found in the sectors

$$B_{pqrs}^{(16-,17-,18-)} = B_{pqrs}^{(1,2,3)} - \alpha. \tag{3.41}$$

Similar states appear in the sectors

$$B_{pqrs}^{(19+,20+,21+)} = B_{pqrs}^{(1,2,3)} + \alpha + z_1, \tag{3.42}$$

and produce the following representations :

- $\{\bar{\eta}^1\}|R >_{pqrs}^{(i)}$, $i = 1, 2, 3$, where $|R >_{pqrs}^{(i)}$ is the degenerated Ramond vacuum of the $B_{pqrs}^{(i)}$ sector. These states transform as a vectorial representation of U(1).
- $\{\bar{\eta}^{2*}\}|R >_{pqrs}^{(i)}$ $and \{\bar{\eta}^{3*}\}|R >_{pqrs}^{(i)}$, $i = 1, 2, 3$. These states transform as a vectorial representation of U(1).
- $\{\bar{\psi}^{1...5}\}|R >_{pqrs}^{(i)}$, $i = 1, 2, 3$. These states transform as a vectorial representation of $SU(5) \times U(1)$.
- $\{\bar{\Phi}^{1..4}\}|R >_{pqrs}^{(i)}$, $i = 1, 2, 3$. These states transform as a vectorial representation of $SU(4) \times U(1)$.

The same states with conjugate oscillators are found in the sectors

$$B_{pqrs}^{(19-,20-,21-)} = B_{pqrs}^{(1,2,3)} - \alpha + z_1. \tag{3.43}$$

Finally in the sectors

$$B_{pqrs}^{(22,23,24)} = B_{pqrs}^{(1,2,3)} + 2\alpha + z_1, \tag{3.44}$$

we find the following representations :

- $\{\bar{\eta}^1\}|R>_{pqrs}^{(i)}$ and $\{\bar{\eta}^{1*}\}|R>_{pqrs}^{(i)}$, $i = 1,2,3$, where $|R>_{pqrs}^{(i)}$ is the degenerated Ramond vacuum of the $B_{pqrs}^{(i)}$ sector. These states transform as a vectorial representation of U(1).

- $\{\bar{\psi}^{1\ldots5}\}|R>_{pqrs}^{(i)}$ and $\{\bar{\psi}^{1\ldots5*}\}|R>_{pqrs}^{(i)}$, $i = 1,2,3$. These states transform as a vectorial representation of $SU(5) \times U(1)$.

- $\{\bar{\Phi}^{1\ldots4}\}|R>_{pqrs}^{(i)}$ and $\{\bar{\Phi}^{1..4*}\}|R>_{pqrs}^{(i)}$, $i = 1,2,3$. These states transform as a vectorial representation of $SU(4) \times U(1)$.

- $\{\bar{\Phi}^5\}|R>_{pqrs}^{(i)}$ and $\{\bar{\Phi}^{5*}\}|R>_{pqrs}^{(i)}$, $i = 1,2,3$. These states transform as a vectorial representation of $U(1)$.

- $\{\bar{\Phi}^{6\ldots8}\}|R>_{pqrs}^{(i)}$ and $\{\bar{\Phi}^{6\ldots8*}\}|R>_{pqrs}^{(i)}$, $i = 1,2,3$. These states transform as a vectorial representation of $SO(6)$.

The projectors for all these states, as well as their matrix formulation, are given in Appendix A.2.

3.4 Perspectives and future work

Now that we have derived algebraic expressions for the projectors, we can analyse the entire massless spectrum for a given choice of GGSO projection coefficients that completely specify a specific string model. These formulae will be inputted into a computer program which can scan the space of string vacua produced by different choices of GGSO coefficients. This work is similar to what has already be done with Pati-Salam models in [7] and is conducted by J.Rizos at the University of Ioannina.

Interesting results will be for example the number of models with three generations and the possibility of models without any exotic massless states with fractional electric charge. This approach will also enable us to pick models with quasi-realistic spectrum, and the next stage will be to analyse the Yukawa couplings and to study phenomenological properties of these models.

Part 4

An $\mathbf{SU(6) \times SU(2)}$ heterotic string model

We now construct a new specific heterotic-string model with an $SU(6) \times SU(2)$ gauge group. It is obtained by enhancement from a model with a Pati-Salam gauge group. We explain the construction and analyse the spectrum, which contains three chiral families and no exotic fractionally charged states. Then we compute the superpotential and derive the F-flatnness and D-flatness conditions.

4.1 Construction and spectrum of an $\mathbf{SU(6) \times SU(2)}$ model

4.1.1 Construction by enhancement from Pati-Salam models

Analysing the possible enhancements from the flipped $SU(5) \times U(1)$ heterotic string model, we observe that the observable gauge group can be enhanced to $SU(6) \times SU(2)$. Since such a model has not been obtained from string theory yet, this enhancement could be interesting . We will therefore study a model with such an observable gauge group.

This enhancement also appeared in the Pati-Salam classification described in [7]. Since a computer program already exists for an analysis of such models, we can use it to obtain the $SU(6) \times SU(2)$ heterotic-string model in the free-fermionic formulation. The construction of this model follows the same steps as for the flipped $SU(5) \times U(1)$ models we described previously. We recall here some of the main features.

Starting with the set of twelve basis vectors (3.2) we add a thirteenth one,

$$\alpha = \{\bar{\psi}^{4,5}, \bar{\Phi}^{1,2}\} \ . \tag{4.1}$$

It breaks the $SO(10)$ symmetry to the $SO(6) \times SO(4)$ Pati-Salam gauge group and the hidden symmetry $SO(8)_1$ to $SO(4)_1 \times SO(4)_2$. The gauge group becomes

$$SO(6) \times SO(4) \times U(1)_1 \times U(1)_2 \times U(1)_3 \times SO(4)_1 \times SO(4)_2 \times SO(8)_2 \ . \tag{4.2}$$

We also define the two vectors

$$x = 1 + S + \sum_{i=1}^{6} e_i + z_1 + z_2 = \{\bar{\eta}^{1,..,3}, \bar{\psi}^{1,..,5}\} \ , \tag{4.3}$$

$$b_3 = b_1 + b_2 + x \ . \tag{4.4}$$

We are interesting in an $SU(6)\times SU(2)$ heterotic string model. This can be realised from the Pati-Salam model, provided that the following enhancement of the observable gauge group from the sector x is realised :

$$SU(4)_{obs} \times SU(2)_R \times U(1)' \longrightarrow SU(6) \times SU(2) \ , \tag{4.5}$$

where $U(1)' = U(1)_1 + U(1)_2 - U(1)_3$. The gauge group in our model is therefore

$$SU(6) \times SU(2) \times U(1)'_1 \times U(1)'_2 \times SO(4)_1 \times SO(4)_2 \times SO(8)_2 \ , \tag{4.6}$$

where

$$U(1)'_1 = 2\,U(1)_1 - U(1)_2 + U(1)_3 \ , \tag{4.7}$$

$$U(1)'_2 = U(1)_2 + U(1)_3 \ , \tag{4.8}$$

are two orthogonal combinations.

As explained in the section 3.4, the work of part 3 can be imputted in a computer program which can span the space of string vacua produced by different choices of GGSO projection coefficients. A similar algorithm is presented in [6, 7] for the Pati-Salam heterotic string model. It allows us to pick a model satisfying certain conditions amongst random sets of GGSO projection coefficients.

We are looking for a model with the following characteristics :

- an $SU(6)\times SU(2)$ observable gauge group,
- three chiral generations for the standard model matter,
- no exotic massless states.

Inputting these conditions in the Pati-Salam program, we obtain the following set of GGSO projection coefficients :

$$(v_i|v_j) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
 & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & & & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
 & & & & & & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
 & & & & & & & 0 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & & & & 1 & 0 & 1 & 1 & 0 \\
 & & & & & & & & & 1 & 0 & 0 & 1 \\
 & & & & & & & & & & 1 & 1 & 1 \\
 & & & & & & & & & & & 1 & 1 \\
 & & & & & & & & & & & & 1
\end{pmatrix} \ , \tag{4.9}$$

where we use the notation $C\binom{v_i}{v_j} = e^{i\pi(v_i|v_j)}$ and where the coefficient under the diagonal are fixed by modular invariance.

This set of coefficients describes a model without massless fractionally charged states. They only exist in the massive spectrum, which is consistent with experimental constraints. Thus this is an exhophobic model.

4.1.2 Matter content of the SU(6) × SU(2) model

We now determine the spectrum of this $SU(6) \times SU(2)$ model. Using the previous notation for the generic sectors, states arise from the following sectors in representations of the Pati-Salam gauge group :

- Neveu-Schwartz (NS) and X sectors : untwisted matter.
- $B^{(1,2,3)}_{pqrs}$ sectors : spinorial twisted matter in the observable sector.
- $B^{(1,2,3)}_{pqrs} + x$ sectors : vectorial twisted matter (observable and hidden).
- $B^{(1,2,3)}_{pqrs} + x + z_1$, $B^{(1,2,3)}_{pqrs} + x + z_2$: spinorial twisted matter in the hidden sector.

Sector	**Field**	$SU(6) \times SU(2)$	$U(1)'_1$	$U(1)'_A$	$SO(4)_1 \times SO(4)_2 \times SO(8)$
$S \oplus S + x$	$\psi^\mu_{\frac{1}{2}}$	(35, 1)	0	0	(1,1,1)
		(1, 3)	0	0	(1,1,1)
		(1, 1)	0	0	(6,1,1)
		(1, 1)	0	0	(1,6,1)
		(1, 1)	0	0	(1,1,28)
		(1, 1)	0	0	(1,1,1)
		(1, 1)	0	0	(1,1,1)
	F_5	(15, 1)	−2	0	(1,1,1)
	$\bar{F}_5$	($\overline{15}$, 1)	2	0	(1,1,1)
	Φ_{12}	(1, 1)	0	2	(1,1,1)
	$\bar{\Phi}_{12}$	(1, 1)	0	−2	(1,1,1)
	Φ_1	(1, 1)	0	0	(1,1,1)
	Φ_2	(1, 1)	0	0	(1,1,1)
	F_6	(15, 1)	1	−1	(1,1,1)
	$\bar{F}_6$	($\overline{15}$, 1)	−1	1	(1,1,1)
	Φ_{34}	(1, 1)	3	1	(1,1,1)
	$\bar{\Phi}_{34}$	(1, 1)	−3	−1	(1,1,1)
	Φ_3	(1, 1)	0	0	(1,1,1)
	Φ_4	(1, 1)	0	0	(1,1,1)
	F_7	(15, 1)	1	1	(1,1,1)
	$\bar{F}_7$	($\overline{15}$, 1)	−1	−1	(1,1,1)
	Φ_{56}	(1, 1)	3	−1	(1,1,1)
	$\bar{\Phi}_{56}$	(1, 1)	−3	1	(1,1,1)
	Φ_5	(1, 1)	0	0	(1,1,1)
	Φ_6	(1, 1)	0	0	(1,1,1)

Table 4.1: Untwisted gauge and matter spectrum

Sector	**Field**	$SU(6) \times SU(2)$	$U(1)'_1$	$U(1)'_A$	$SO(4)_1 \times SO(4)_2 \times SO(8)$
$S + b_1 + e_4 \oplus S + b_1 + e_4 + x$	F_1	$(15, 1)$	1	0	$(1, 1, 1)$
	χ_1	$(1, 1)$	-3	0	$(1, 1, 1)$
	ζ_1	$(1, 1)$	0	1	$(1, 1, 1)$
	ζ_2	$(1, 1)$	0	1	$(1, 1, 1)$
	$\bar{\zeta}_1$	$(1, 1)$	0	-1	$(1, 1, 1)$
	$\bar{\zeta}_2$	$(1, 1)$	0	-1	$(1, 1, 1)$
$S + b_1 + e_4 + e_6 \oplus S + b_1 + e_4 + e_6 + x$	$\bar{f}_1$	$(\bar{6}, 2)$	1	0	$(1, 1, 1)$
$S + b_2 + e_2 \oplus S + b_2 + e_2 + x$	F_2	$(15, 1)$	$-\frac{1}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	χ_2	$(1, 1)$	$\frac{3}{2}$	$-\frac{3}{2}$	$(1, 1, 1)$
	ζ_4	$(1, 1)$	$\frac{3}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	ζ_3	$(1, 1)$	$\frac{3}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	$\bar{\zeta}_3$	$(1, 1)$	$-\frac{3}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
	$\bar{\zeta}_4$	$(1, 1)$	$-\frac{3}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
$S + b_2 + e_2 + e_6 \oplus S + b_2 + e_2 + e_6 + x$	$\bar{f}_2$	$(\bar{6}, 2)$	$-\frac{1}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
$S + b_2 + e_1 + e_6 \oplus S + b_2 + e_1 + e_6 + x$	F_3	$(15, 1)$	$-\frac{1}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	χ_3	$(1, 1)$	$\frac{3}{2}$	$-\frac{3}{2}$	$(1, 1, 1)$
	ζ_5	$(1, 1)$	$\frac{3}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	$\bar{\zeta}_6$	$(1, 1)$	$-\frac{3}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
	ζ_6	$(1, 1)$	$\frac{3}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	$\bar{\zeta}_5$	$(1, 1)$	$-\frac{3}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
$S + b_2 + e_1 \oplus S + b_2 + e_1 + x$	$\bar{f}_3$	$(\bar{6}, 2)$	$-\frac{1}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
$S + b_2 + x + e_1 + e_2$	ζ_{11}	$(1, 1)$	$\frac{3}{2}$	$\frac{1}{2}$	$(1,1,1)$
	$\bar{\zeta}_{11}$	$(1, 1)$	$-\frac{3}{2}$	$-\frac{1}{2}$	$(1,1,1)$
$S + b_2 + x + e_6$	ζ_{12}	$(1, 1)$	$\frac{3}{2}$	$\frac{1}{2}$	$(1,1,1)$
	$\bar{\zeta}_{12}$	$(1, 1)$	$-\frac{3}{2}$	$-\frac{1}{2}$	$(1,1,1)$
$S + b_3 + e_4 + e_2 \oplus S + b_3 + e_4 + e_2 + x$	F_4	$(15, 1)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
	χ_4	$(1, 1)$	$+\frac{3}{2}$	$+\frac{3}{2}$	$(1, 1, 1)$
	ζ_8	$(1, 1)$	$\frac{3}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
	$\bar{\zeta}_8$	$(1, 1)$	$-\frac{3}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	ζ_7	$(1, 1)$	$\frac{3}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
	$\bar{\zeta}_7$	$(1, 1)$	$-\frac{3}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
$S + b_3 + e_4 + e_2 + e_1 \oplus S + b_3 + e_4 + e_2 + e_1 + x$	$\bar{F}_4$	$(\overline{15}, 1)$	$\frac{1}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	χ_5	$(1, 1)$	$-\frac{3}{2}$	$-\frac{3}{2}$	$(1, 1, 1)$
	ζ_9	$(1, 1)$	$-\frac{3}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	ζ_{10}	$(1, 1)$	$-\frac{3}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
	$\bar{\zeta}_{10}$	$(1, 1)$	$\frac{3}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
	$\bar{\zeta}_9$	$(1, 1)$	$\frac{3}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$
$S + b_3 + e_4 \oplus S + b_3 + e_4 + x$	f_1	$(6, 2)$	$\frac{1}{2}$	$\frac{1}{2}$	$(1, 1, 1)$
$S + b_3 + e_4 + e_1 \oplus S + b_3 + e_4 + e_1 + x$	$\bar{f}_4$	$(\bar{6}, 2)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$(1, 1, 1)$

Table 4.2: Observable twisted matter spectrum

Sector	**Field**	$SU(6)\times SU(2)$	$U(1)'_1$	$U(1)'_A$	$SO(4)_1\times SO(4)_2\times SO(8)$
$S+b_1+x+e_4+e_5$	H_{11}	(1,1)	0	1	(4,1,1)
$S+b_1+x+e_4+e_5+e_6$	H_{21}	(1,1)	0	1	(1,4,1)
$S+b_1+x+e_3+e_4+e_6$	Z_1	(1,1)	0	1	(1,1,8)
$S+b_2+x+e_1+e_2+e_5$	H_{12}	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$	(4,1,1)
$S+b_2+x+e_1+e_2+e_5+e_6$	H_{22}	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$	(1,4,1)
$S+b_2+x+e_5+e_6$	H_{13}	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$	(4,1,1)
$S+b_2+x+e_5$	H_{23}	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$	(1,4,1)
$S+b_3+x+e_1+e_3+e_4$	Z_2	(1,1)	$-\frac{3}{2}$	$\frac{1}{2}$	(1,1,8)
$S+b_3+x+e_3+e_4$	Z_3	(1,1)	$\frac{3}{2}$	$-\frac{1}{2}$	(1,1,8)
$S+b_1+x+z_2+e_3+e_4+e_5+e_6$	Z_4	(1,1)	0	1	(1,1,8)
$S+b_1+x+z_2+e_4+e_5$	Z_5	(1,1)	0	1	(1,1,8)
$S+b_1+x+z_1+e_3+e_4+e_5+e_6$	H_{121}	(1,1)	0	1	$(\bar{2},\bar{2},1)$
$S+b_1+x+z_1+e_3+e_4+e_5$	H_{122}	(1,1)	0	1	$(2,2,1)$
$S+b_1+x+z_1+e_3+e_4+e_6$	H_{123}	(1,1)	0	-1	$(2,\bar{2},1)$
$S+b_1+x+z_1+e_3+e_4$	H_{124}	(1,1)	0	-1	$(\bar{2},2,1)$
$S+b_3+x+z_1$	H_{125}	(1,1)	$-\frac{3}{2}$	$\frac{1}{2}$	$(\bar{2},\bar{2},1)$
$S+b_3+x+z_1+e_2$	H_{126}	(1,1)	$-\frac{3}{2}$	$\frac{1}{2}$	$(2,2,1)$
$S+b_3+x+z_1+e_1$	H_{127}	(1,1)	$-\frac{3}{2}$	$\frac{1}{2}$	$(\bar{2},\bar{2},1)$
$S+b_3+x+z_1+e_2+e_1$	H_{128}	(1,1)	$-\frac{3}{2}$	$\frac{1}{2}$	$(2,2,1)$

Table 4.3: Twisted hidden matter spectrum.

Because of the enhancement the states arising from the observable sector fall in representations of the $SU(6)\times SU(2)$ gauge group. These can be obtained by adding the states in a sector to the states in this sector plus x, for example $b_i \oplus b_i + x$.

The gauge bosons come from the sector $NS \oplus NS{+}x$ in the representation $(\mathbf{35},\mathbf{1})+(\mathbf{1},\mathbf{3})$, according to the decomposition of the adjoint of $SU(6)$ under $SU(4)\times SU(2)\times U(1)$:

$$\mathbf{35} = (\mathbf{4},\mathbf{2},-3)+(\bar{\mathbf{4}},\mathbf{2},3)+(\mathbf{15},\mathbf{1},0)+(\mathbf{1},\mathbf{3},0)+(\mathbf{1},\mathbf{1},0)\ . \tag{4.10}$$

The observable matter spectrum can be accomodated in the representation $\mathbf{27}$ of E_6. Therefore according to the embedding

$$\mathbf{27} = (\mathbf{15},\mathbf{1})+(\bar{\mathbf{6}},\mathbf{2})\ , \tag{4.11}$$

the observable matter spectrum must be in the representations $(\mathbf{15},\mathbf{1})$ and $(\bar{\mathbf{6}},\mathbf{2})$ of $SU(6)\times SU(2)_R$.

Decomposing these representations under Pati Salam $SU(4)\times SU(2)\times SU(2)$ gauge group gives

$$(\mathbf{15},\mathbf{1}) = (\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{6},\mathbf{1},\mathbf{1})+(\mathbf{4},\mathbf{2},\mathbf{1})\ , \tag{4.12}$$

$$(\bar{\mathbf{6}},\mathbf{2}) = (\mathbf{1},\mathbf{2},\mathbf{2})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})\ . \tag{4.13}$$

Then the standard model is embedded in the Pati-Salam gauge group in the following way :

$$\begin{cases} (4,2,1) = \left(3,2,\frac{1}{6}\right)_Q + \left(1,2,-\frac{1}{2}\right)_L , \\ (\bar{4},1,2) = \left(\bar{3},1,-\frac{2}{3}\right)_{u^c} + \left(\bar{3},1,\frac{1}{3}\right)_{d^c} + (1,1,1)_{e^c} + (1,1,0)_{\nu^c} , \\ (1,2,2) = \left(1,2,-\frac{1}{2}\right)_h + \left(1,2,\frac{1}{2}\right)_{h^c} , \\ (6,1,1) = \left(3,1,-\frac{1}{3}\right)_D + \left(\bar{3},1,\frac{1}{3}\right)_{D^c} , \\ (1,1,1) = (1,1,0)_N . \end{cases} \tag{4.14}$$

We thus find the quarks and leptons of the standard model. We also have two Higgs h and h^c in the representation $(1,2,2)$ of Pati-Salam which are used to break the electroweak symmetry and two coloured triplets D and D^c.

The full massless spectrum is shown in tables 4.1, 4.2 and 4.3. It contains :

- The quarks and leptons fields in three chiral generations from $(\mathbf{15},1)+(\bar{\mathbf{6}},2)$.
- A pair of Higgs in $(\mathbf{15},1)+(\bar{\mathbf{15}},1)$ to break the $SU(6)\times SU(2)_R$ symmetry.
- A pair of Higgs in $(\mathbf{6},2)+(\bar{\mathbf{6}},2)$ to break the Pati-Salam symmetry.
- Light Higgs doublets in $(\mathbf{6},2)+(\bar{\mathbf{6}},2)$ to break the electroweak symmetry.
- Hidden matter which transforms according to representations of the hidden gauge group.

$\mathbf{U(1)}$ anomaly

Looking at the charges of the states, it appears that this model contains one anomalous $U(1)$:

$$U(1)'_A = U(1)'_2 \quad \mathrm{Tr}\, U(1)'_A = 72 , \tag{4.15}$$

while $U(1)'_1$ is non anomalous.

We will see later that this anomaly can be cancelled using the Dine-Seiberg-Witten mechanism.

4.2 Analysis of the model

4.2.1 Derivation of the superpotential

We now calculate the cubic-level superpotential, following the method described by A.E. Faraggi in [10]. We will not explain the rules in detail since a complete understanding of how they are derived is not necessary for the following work.

The trilevel superpotential of our model describes the Yukawa couplings and thus enables to compute the mass of particles that interact with Higgs fields.

It is obtained by calculating correlators between vertex operators $V_i^{f(b)}$:

$$A_3 = \left\langle \int d^2z_1 \int d^2z_2 \int d^2z_3 \; V_1^f V_2^f V_3^b \right\rangle \; . \tag{4.16}$$

Each term in the superpotential will be the product of three-point functions corresponding to different world-sheet fields. The non vanishing terms in the trilevel superpotential are those for which the following rules are respected :

1. For a complexified two-dimensional fermion which produces a U(1) current, the total $U(1)$-charge must be zero : $Q_1 + Q_2 + Q_3 = 0$.
   This concerns the fields $\{\chi^{12}, \chi^{34}, \chi^{56}, \bar{\psi}^{1..5}, \bar{\eta}^{1..3}, \bar{\Phi}^{1..8}\}$.

2. A left-moving real two-dimensional fermion f, paired with a right-moving real fermion $\bar{f}$, produces an Ising model operator $\sigma_\pm$, corresponding to the two different fermion number projections in the Ramond sector. Thus real fermions, *i.e.* y^i and w^i, can only appear in the following combinations :

   – $\langle f(z_1)\sigma_\pm(z_2)\rangle$,
   – $\langle \sigma_\pm(z_1)\sigma_\pm(z_2)\rangle$,
   – $\langle \sigma_+(z_1)\sigma_-(z_2)f(z_3)\rangle$,
   – $\langle \sigma_+(z_1)\sigma_-(z_2)\bar{f}(z_3)\rangle$.

To find the non vanishing terms in the superpotential, we have to check the previous rules for all the fermions in each possible term.

The superpotential for the $SU(6) \times SU(2)$ heterotic string model is then :

$$\begin{aligned}\frac{W_{SM}}{g\sqrt{2}} =& \{ F_1 F_2 F_4 + \bar{f}_1 \bar{f}_2 F_4 \} + \{ \bar{f}_3 \bar{f}_4 F_1 + \bar{f}_4 f_1 F_3 \} + \{ F_1 F_1 F_5 + F_2 F_2 F_6 \\ &+ F_4 F_4 F_7 + \bar{f}_1 \bar{f}_1 F_5 + \bar{f}_2 \bar{f}_2 F_6 + \bar{f}_4 \bar{f}_4 F_7 \} + \{ F_3 F_3 F_6 + \bar{f}_3 \bar{f}_3 F_6 \\ &+ f_1 f_1 \bar{F}_7 + \bar{F}_4 \bar{F}_4 \bar{F}_7 \} + \{ \bar{F}_5 F_1 \chi_1 + \bar{F}_6 F_2 \chi_2 + \bar{F}_6 F_3 \chi_3 + \bar{F}_7 F_4 \chi_4 \\ &+ F_7 \bar{F}_4 \chi_5 \} + \chi_1 \chi_2 \chi_4 + \{ F_5 F_6 F_7 + \bar{F}_5 \bar{F}_6 \bar{F}_7 \} + \{ \Phi_{12} \bar{\Phi}_{34} \Phi_{56} + \bar{\Phi}_{12} \Phi_{34} \bar{\Phi}_{56} \} \\ &+ \{ \bar{F}_5 F_6 \bar{\Phi}_{56} + \bar{F}_5 F_7 \bar{\Phi}_{34} + F_5 \bar{F}_6 \Phi_{56} + F_5 \bar{F}_7 \Phi_{34} + \bar{F}_6 F_7 \bar{\Phi}_{12} + \bar{F}_7 F_6 \Phi_{12} \} \\ &+ \{ \zeta_2 \bar{\zeta}_7 \chi_2 + \zeta_1 \bar{\zeta}_8 \chi_2 + \bar{\zeta}_1 \bar{\zeta}_3 \chi_4 + \bar{\zeta}_2 \bar{\zeta}_4 \chi_4 + \zeta_3 \zeta_7 \chi_1 + \zeta_4 \zeta_8 \chi_1 \\ &+ \bar{\zeta}_{10} \zeta_{11} \chi_1 + \zeta_1 \zeta_{11} \chi_5 \} \\ &+ \{ H_{11} H_{11} + H_{21} H_{21} + H_{121} H_{121} + H_{122} H_{122} + Z_1 Z_1 + Z_4 Z_4 + Z_5 Z_5 \} \bar{\Phi}_{12} \\ &+ \{ H_{12} H_{12} + H_{22} H_{22} + H_{13} H_{13} + H_{23} H_{23} \} \bar{\Phi}_{34} \\ &+ \{ H_{125} H_{125} + H_{126} H_{126} + H_{127} H_{127} + H_{128} H_{128} + Z_2 Z_2 \} \Phi_{56} \\ &+ \{ H_{123} H_{123} + H_{124} H_{124} \} \Phi_{12} + Z_3 Z_3 \bar{\Phi}_{56} + \{ H_{11} H_{12} + H_{21} H_{22} \} \chi_5 \\ &+ Z_1 Z_2 \chi_3 + \frac{1}{\sqrt{2}} Z_1 Z_3 \bar{\zeta}_{12} + \{ \zeta_1 \zeta_1 + \zeta_2 \zeta_2 \} \bar{\Phi}_{12} + \{ \bar{\zeta}_1 \bar{\zeta}_1 + \bar{\zeta}_2 \bar{\zeta}_2 \} \Phi_{12} \\ &+ \{ \zeta_4 \zeta_4 + \zeta_3 \zeta_3 + \zeta_5 \zeta_5 + \zeta_6 \zeta_6 + \zeta_{11} \zeta_{11} + \zeta_{12} \zeta_{12} \} \bar{\Phi}_{34} + \{ \bar{\zeta}_3 \bar{\zeta}_3 + \bar{\zeta}_4 \bar{\zeta}_4 \\ &+ \bar{\zeta}_6 \bar{\zeta}_6 + \bar{\zeta}_5 \bar{\zeta}_5 + \bar{\zeta}_{11} \bar{\zeta}_{11} + \bar{\zeta}_{12} \bar{\zeta}_{12} \} \Phi_{34} + \{ \zeta_8 \zeta_8 + \zeta_7 \zeta_7 + \bar{\zeta}_{10} \bar{\zeta}_{10} + \bar{\zeta}_9 \bar{\zeta}_9 \} \bar{\Phi}_{56} \\ &+ \{ \bar{\zeta}_8 \bar{\zeta}_8 + \bar{\zeta}_7 \bar{\zeta}_7 + \zeta_9 \zeta_9 + \zeta_{10} \zeta_{10} \} \Phi_{56} + \{ \zeta_{11} \bar{\zeta}_{11} + \zeta_{12} \bar{\zeta}_{12} \} \Phi_4 \\ &+ \frac{1}{\sqrt{2}} \{ F_1 \bar{F}_4 \bar{\zeta}_{11} + \bar{f}_1 f_1 \bar{\zeta}_{12} + \zeta_1 \bar{\zeta}_{10} \bar{\zeta}_{11} + \zeta_2 \bar{\zeta}_9 \bar{\zeta}_{11} \} \ .\end{aligned} \tag{4.17}$$

4.2.2 Anomaly cancellation

We now discuss the pattern of symmetry breaking and the anomaly cancellation. Once again, we will only describe the procedure without trying to have a profound understanding of how it works.

In order to cancel the U(1) anomaly, we have to define a flat direction. It is a (complex) scalar field whose potential vanishes along that direction :

$$\text{(D-flatness)} : \quad D_a = \sum_i Q_i^{(a)} |\varphi_i|^2 = 0 \ , \tag{4.18}$$

$$\text{(F-flatness)} : \quad F_i = \frac{\partial W}{\partial \eta_i} \ ; \ \ W = 0 \ . \tag{4.19}$$

The anomalous $U(1)'_A$ is broken by the Green-Schwartz-Dine-Seiberg-Witten mechanism in which a potentially large Fayet-Iliopoulos D-term $\xi = \frac{g}{192\pi^2}\mathrm{Tr}U(1)'_A$ is generated by the vacuum expectation value (VEV) of the dilaton field. Such a term would in general break supersymmetry unless there exists a direction in the scalar potential $\Phi = \sum_i \alpha_i \Phi_i$ which is F-flat and D-flat with respect to the non anomalous symetries. In that case this direction will acquire a VEV, cancelling the ξ-term, restauring supersymmetry and stabilizing the vacuum.

In concrete terms, we impose a Fayet-Illiopoulos D-term in the anomalous direction, giving non zero VEVs to certain massless fields :

$$D_A = \sum_i Q_i^{(A)} |\varphi_i|^2 + \frac{g}{192\pi^2}\mathrm{Tr}U(1)'_A = 0 \tag{4.20}$$

while D-flatness in the non anomalous direction and F-flatness are maintained.

We now write the D-flatness and F-flatness constraints for the $SU(6)\times SU(2)$ model.

D-flatness constraints

$$\begin{aligned} U(1)'_1\colon\ D_1 = &-6|\chi_1|^2+3|\chi_2|^2+3|\chi_3|^2+3|\chi_4|^2-3|\chi_5|^2+3(|\zeta_4|^2+3|\zeta_3|^2-3|\bar\zeta_3|^2-3|\bar\zeta_4|^2)\\ &+3(|\zeta_5|^2-3|\bar\zeta_6|^2+3|\zeta_6|^2-3|\bar\zeta_5|^2)+3(|\zeta_8|^2-3|\bar\zeta_8|^2+3|\zeta_7|^2-3|\bar\zeta_7|^2)\\ &+3(|\bar\zeta_9|^2+|\bar\zeta_{10}|^2-3|\zeta_9|^2-3|\zeta_{10}|^2)+3(|\zeta_{11}|^2-3|\bar\zeta_{11}|^2+3|\zeta_{12}|^2-3|\bar\zeta_{12}|^2)\\ &+6(|\Phi_2|^2-6|\bar\Phi_2|^2)+6(|\Phi_3|^2-6|\bar\Phi_3|^2)\\ &+2|f_1|^2-|\bar f_2|^2-|\bar f_3|^2+|f_1|^2-|\bar f_4|^2\\ &+2|F_1|^2-|F_2|^2-|F_3|^2-|F_4|^2+|\bar F_4|^2\ =\ 0\ , \end{aligned} \tag{4.21}$$

$$\begin{aligned} U(1)'_A\colon\ D_A = &-3|\chi_2|^2-3|\chi_3|^2+3|\chi_4|^2-3|\chi_5|^2+2(|\zeta_1|^2+2|\zeta_2|^2-2|\bar\zeta_1|^2-2|\bar\zeta_2|^2)\\ &+(|\zeta_4|^2+|\zeta_3|^2-|\bar\zeta_3|^2-|\bar\zeta_4|^2)+(|\zeta_5|^2-|\bar\zeta_6|^2+|\zeta_6|^2-|\bar\zeta_5|^2)\\ &+(|\bar\zeta_7|^2+|\bar\zeta_8|^2-|\zeta_7|^2-|\zeta_8|^2)+(|\zeta_9|^2+|\zeta_{10}|^2-|\bar\zeta_{10}|^2-|\bar\zeta_9|^2)\\ &+(|\zeta_{11}|^2-|\bar\zeta_{11}|^2+|\zeta_{12}|^2-|\bar\zeta_{12}|^2)+4(|\Phi_1|^2-|\bar\Phi_1|^2)+2(|\Phi_2|^2-|\bar\Phi_2|^2)\\ &+2(|\bar\Phi_3|^2-|\Phi_3|^2)+|\bar f_2|^2+|\bar f_3|^2+|f_1|^2-|\bar f_4|^2\\ &+|F_2|^2+2|F_3|^2-|F_4|^2+|\bar F_4|^2+\ \frac{3\,g^2_{\rm string}\,M_P^2}{8\pi^2}\ =\ 0\ . \end{aligned} \tag{4.22}$$

F-flatness constraint

$$\frac{\partial W}{\partial \eta_i}=0\ \ \text{for all fields } \eta_i\ ;\ \ W=0\ . \tag{4.23}$$

It gives a set of non-linear equations which are shown in the Appendix B.

Additional conditions

The next step is to solve these equations. For that purpose we need to add additionnal constraints. The first one is that the Higgs doublets in $(\mathbf{6},\mathbf{2})+(\bar{\mathbf{6}},\mathbf{2})$ must remain light. They correspond to the fields $\bar f_1,\ \bar f_2,\ \bar f_3,\ \bar f_4$ which are in $(\bar{\mathbf{6}},\mathbf{2})$ and f_1 which is in $(\mathbf{6},\mathbf{2})$. We write down the Higgs doublets mass matrix

$$M_f=\begin{pmatrix} F_5 & F_4 & 0 & \bar\zeta_{12} & F_3\\ F_4 & F_6 & 0 & 0 & 0\\ 0 & 0 & F_6 & 0 & F_1\\ \bar\zeta_{12} & 0 & 0 & \bar F_7 & 0\\ F_3 & 0 & F_1 & 0 & F_7 \end{pmatrix}\ . \tag{4.24}$$

Its determinant must vanish to ensure that the electroweak Higgs fields are massless :

$$\det(M_f)=[(F_5F_6-F_4{}^2)\bar F_7-\bar\zeta_{12}^2F_6](F_6F_7-F_1^2)-F_3{}^2F_6^2\bar F_7=0\ . \tag{4.25}$$

In order to break $SU(6)$, we choose a pair of $(15,1)+(\bar{15},1)$, namely F_3 and $\bar{F}_4$. They must be massless too, which is ensured by the condition $F_6\bar{F}_7 = 0$.

Then in order to break Pati-Salam, we also have to choose a pair of $(6,2)+(\bar{6},2)$, for example $\bar{f}_2$ and f_1, which are also massless.

We can now present a solution, assuming that the fields $F_1,\ F_2,\ F_4,\ F_5,\ F_6,\ F_7,\ \bar{F}_5,$ $\bar{F}_6,\ \bar{F}_7,\ \bar{f}_1,\ \bar{f}_3,\ \bar{f}_4$ don't have a VEV.

The minimal choice

$$F_3 = \bar{F}_4 \tag{4.26}$$

$$\bar{f}_2 = f_1 \tag{4.27}$$

$$|\bar{\Phi}_{12}|^2 = +\frac{1}{2}|\bar{f}_2|^2 + \frac{3}{4}|F_3|^2 + \frac{3\,g^2_{\text{string}}\,M_P^2}{24\pi^2} \tag{4.28}$$

with all the other field VEVs vanishing, satisfies all F and D-flatness conditions.

We have described an exophobic $SU(6)\times SU(2)$ heterotic string model which contains the Higgs representations that are needed to break the $SU(6)\times SU(2)$ and Pati-Salam symmetries and to generate fermion masses at the electroweak scale. It also contains the three generations of the Standard Model, as expected. We have also calculated the superpotential and found F- and D-flat directions which leave the Higgs doublets light.

Conclusion

We have developed tools to analyse the spectrum of a large class of models in which the $SO(10)$ GUT symmetry is broken to the flipped $SU(5) \times U(1)$ subgroup. We have derived algebraic expressions that allow us to compute easily the gauge group and the matter content of these models. Once these expressions are entered in a computer program, a classification of these models will provide us with a more accurate knowledge of the string vacua. In particular one interesting feature will be the number of models with three generations of chiral matter. We would also like to know whether such models exist such models without exotic fractionally charged states.

Another part of this work has been the construction of the first $SU(6) \times SU(2)$ string model. This model has been obtained from a similar classification of heterotic string vacua with a Pati-Salam gauge group and contains three generations of chiral matter and no exotic fractionally charged states. It also contains the Higgs representations needed to break $SU(6) \times SU(2)$ and the Pati-Salam down to the Standard Model. The next step in this work will be to look at some phenomenological properties of the model, such as the masses of the particles of the three generations.

This new approach consists in classifying string vacua instead of picking up isolated models. It allows either to scan a large class of models, which is what we have developed for flipped $SU(5) \times U(1)$ models, or to select a single model with interesting properties from the classification and to analyse it, as we have done with the $SU(6) \times SU(2)$ model. For now it remains a mystery how a specific string vacuum can be dynamically singled out. This is matter for future work.

Appendices

A. Projectors and matrix formalism

A.1 Spinorial representations

Sectors $B_{pqrs}^{(4\pm)}$, $B_{pqrs}^{(5\pm)}$, $B_{pqrs}^{(6\pm)}$

$$B_{pqrs}^{(4+,5+,6+)} = B_{pqrs}^{(1,2,3)} + \alpha + z_2 \tag{4.29}$$
$$B_{pqrs}^{(4-,5-,6-)} = B_{pqrs}^{(1,2,3)} - \alpha + z_2 \tag{4.30}$$

$$\begin{aligned} P_{pqrs}^{((3+i)\pm)} \quad = \quad \tfrac{1}{16} \quad & \left(1 - C\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{((3+i)\pm)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B_{pqrs}^{((3+i)\pm)} \end{pmatrix}\right) \\ & \cdot \left(1 + C\begin{pmatrix} z_1 \\ B_{pqrs}^{((3+i)\pm)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} \alpha \\ B_{pqrs}^{((3+i)\pm)} \end{pmatrix}\right) \end{aligned} \tag{4.31}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 \pm \alpha + z_2) \\ (e_2|b_1 \pm \alpha + z_2) \\ 1 + (z_1|b_1 \pm \alpha + z_2) \\ (\alpha|b_1 \pm \alpha + z_2) \end{pmatrix} \tag{4.32}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 \pm \alpha + z_2) \\ (e_4|b_2 \pm \alpha + z_2) \\ 1 + (z_1|b_2 \pm \alpha + z_2) \\ (\alpha|b_2 \pm \alpha + z_2) \end{pmatrix} \tag{4.33}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_3) & (\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3 \pm \alpha + z_2) \\ (e_6|b_3 \pm \alpha + z_2) \\ 1 + (z_1|b_3 \pm \alpha + z_2) \\ (\alpha|b_3 \pm \alpha + z_2) \end{pmatrix} \tag{4.34}$$

Sectors $B^{(7\pm)}_{pqrs}$, $B^{(8\pm)}_{pqrs}$, $B^{(9\pm)}_{pqrs}$

$$B^{(7+,8+,9+)}_{pqrs} = B^{(1,2,3)}_{pqrs} + \alpha + z_1 + z_2 \tag{4.35}$$
$$B^{(7-,8-,9-)}_{pqrs} = B^{(1,2,3)}_{pqrs} - \alpha + z_1 + z_2 \tag{4.36}$$

$$\begin{aligned} P^{((6+i)\pm)}_{pqrs} = \tfrac{1}{16} & \left(1 - C\begin{pmatrix} e_{2i-1} \\ B^{((6+i)\pm)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B^{((6+i)\pm)}_{pqrs} \end{pmatrix}\right) \\ & \cdot \left(1 + C\begin{pmatrix} z_1 \\ B^{((6+i)\pm)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 + C\begin{pmatrix} \alpha \\ B^{((6+i)\pm)}_{pqrs} \end{pmatrix}\right) \end{aligned} \tag{4.37}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 \pm \alpha + z_1 + z_2) \\ (e_2|b_1 \pm \alpha + z_1 + z_2) \\ 1 + (z_1|b_1 \pm \alpha + z_1 + z_2) \\ 1 + (\alpha|b_1 \pm \alpha + z_1 + z_2) \end{pmatrix} \tag{4.38}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 \pm \alpha + z_1 + z_2) \\ (e_4|b_2 \pm \alpha + z_1 + z_2) \\ 1 + (z_1|b_2 \pm \alpha + z_1 + z_2) \\ 1 + (\alpha|b_2 \pm \alpha + z_1 + z_2) \end{pmatrix} \tag{4.39}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_3) & (\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3 \pm \alpha + z_1 + z_2) \\ (e_6|b_3 \pm \alpha + z_1 + z_2) \\ 1 + (z_1|b_3 \pm \alpha + z_1 + z_2) \\ 1 + (\alpha|b_3 \pm \alpha + z_1 + z_2) \end{pmatrix} \tag{4.40}$$

Sectors $B^{(10)}_{pqrs}$, $B^{(11)}_{pqrs}$, $B^{(12)}_{pqrs}$

$$B^{(10,11,12)}_{pqrs} = B^{(1,2,3)}_{pqrs} + 2\alpha \tag{4.41}$$

$$P^{(9+i)}_{pqrs} = \frac{1}{8}\left(1 - C\begin{pmatrix} e_{2i-1} \\ B^{(9+i)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B^{(9+i)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} z_2 \\ B^{(9+i)}_{pqrs} \end{pmatrix}\right) \tag{4.42}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+2\alpha) \\ (e_2|b_1+2\alpha) \\ (z_2|b_1+2\alpha) \end{pmatrix} \tag{4.43}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+2\alpha) \\ (e_4|b_2+2\alpha) \\ (z_2|b_2+2\alpha) \end{pmatrix} \tag{4.44}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+2\alpha) \\ (e_6|b_3+2\alpha) \\ (z_2|b_3+2\alpha) \end{pmatrix} \tag{4.45}$$

Sectors $B^{(13)}_{pqrs}$, $B^{(14)}_{pqrs}$, $B^{(15)}_{pqrs}$

$$B^{(13,14,15)}_{pqrs} = B^{(1,2,3)}_{pqrs} + 2\alpha + z_1 + z_2 \tag{4.46}$$

$$P^{(12+i)}_{pqrs} = \frac{1}{8}\left(1 - C\begin{pmatrix} e_{2i-1} \\ B^{(12+i)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B^{(12+i)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} z_1 \\ B^{(12+i)}_{pqrs} \end{pmatrix}\right) \tag{4.47}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+2\alpha) \\ (e_2|b_1+2\alpha) \\ (z_1|b_1+2\alpha) \end{pmatrix} \tag{4.48}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+2\alpha) \\ (e_4|b_2+2\alpha) \\ (z_1|b_2+2\alpha) \end{pmatrix} \tag{4.49}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+2\alpha) \\ (e_6|b_3+2\alpha) \\ (z_1|b_3+2\alpha) \end{pmatrix} \tag{4.50}$$

A.2 Vectorial representations

Sectors $B_{pqrs}^{(16+)}$, $B_{pqrs}^{(17+)}$, $B_{pqrs}^{(18+)}$

$$B_{pqrs}^{(16+,17+,18+)} = B_{pqrs}^{(1,2,3)} + \alpha \tag{4.51}$$

States $\{\bar{\eta}^1\}|R>$ and $\{\bar{\psi}^{1..5}\}|R>$

$$\begin{aligned} P_{pqrs}^{((15+i)+)(\bar{\eta}^1;\bar{\psi}^{1..5})} = \tfrac{1}{16} \quad & \left(1 - C\binom{e_{2i-1}}{B_{pqrs}^{((15+i)+)}}\right) \cdot \left(1 - C\binom{e_{2i}}{B_{pqrs}^{((15+i)+)}}\right) \\ & \cdot \left(1 + C\binom{z_1}{B_{pqrs}^{((15+i)+)}}\right) \cdot \left(1 - i\, C\binom{\alpha + z_2}{B_{pqrs}^{((15+i)+)}}\right) \end{aligned} \tag{4.52}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2 + \alpha|e_3) & (z_2 + \alpha|e_4) & (z_2 + \alpha|e_5) & (z_2 + \alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + \alpha) \\ (e_2|b_1 + \alpha) \\ 1 + (z_1|b_1 + \alpha) \\ 1/2 + (\alpha + z_2|b_1 + \alpha) \end{pmatrix} \tag{4.53}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2 + \alpha|e_1) & (z_2 + \alpha|e_2) & (z_2 + \alpha|e_5) & (z_2 + \alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + \alpha) \\ (e_4|b_2 + \alpha) \\ 1 + (z_1|b_2 + \alpha) \\ 1/2 + (\alpha + z_2|b_2 + \alpha) \end{pmatrix} \tag{4.54}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2 + \alpha|e_1) & (z_2 + \alpha|e_2) & (z_2 + \alpha|e_3) & (z_2 + \alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3 + \alpha) \\ (e_6|b_3 + \alpha) \\ 1 + (z_1|b_3 + \alpha) \\ 1/2 + (\alpha + z_2|b_3 + \alpha) \end{pmatrix} \tag{4.55}$$

States $\{\bar{\eta}^{2*}\}|R>$ and $\{\bar{\eta}^{3*}\}|R>$

$$\begin{aligned} P_{pqrs}^{((15+i)+)(\bar{\eta}^{2*};\bar{\eta}^{3*})} = \frac{1}{16} \quad & \left(1 - C\binom{e_{2i-1}}{B_{pqrs}^{((15+i)+)}}\right) \cdot \left(1 - C\binom{e_{2i}}{B_{pqrs}^{((15+i)+)}}\right) \\ & \cdot \left(1 + C\binom{z_1}{B_{pqrs}^{((15+i)+)}}\right) \cdot \left(1 + i\, C\binom{\alpha + z_2}{B_{pqrs}^{((15+i)+)}}\right) \end{aligned} \tag{4.56}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha) \\ (e_2|b_1+\alpha) \\ 1+(z_1|b_1+\alpha) \\ -1/2+(\alpha+z_2|b_1+\alpha) \end{pmatrix} \tag{4.57}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+\alpha) \\ (e_4|b_2+\alpha) \\ 1+(z_1|b_2+\alpha) \\ -1/2+(\alpha+z_2|b_2+\alpha) \end{pmatrix} \tag{4.58}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+\alpha) \\ (e_6|b_3+\alpha) \\ 1+(z_1|b_3+\alpha) \\ -1/2+(\alpha+z_2|b_3+\alpha) \end{pmatrix} \tag{4.59}$$

States $\{\bar{\varphi}^{1..4*}\}|R>$

$$\begin{aligned} P_{pqrs}^{((15+i)+)(\bar{\varphi}^{1..4*})} = \tfrac{1}{16} & \left(1 - C\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{((15+i)+)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B_{pqrs}^{((15+i)+)} \end{pmatrix}\right) \\ & \cdot \left(1 - C\begin{pmatrix} z_1 \\ B_{pqrs}^{((15+i)+)} \end{pmatrix}\right) \cdot \left(1 + i\, C\begin{pmatrix} \alpha + z_2 \\ B_{pqrs}^{((15+i)+)} \end{pmatrix}\right) \end{aligned} \tag{4.60}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha) \\ (e_2|b_1+\alpha) \\ (z_1|b_1+\alpha) \\ -1/2+(\alpha+z_2|b_1+\alpha) \end{pmatrix} \tag{4.61}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+\alpha) \\ (e_4|b_2+\alpha) \\ (z_1|b_2+\alpha) \\ -1/2+(\alpha+z_2|b_2+\alpha) \end{pmatrix} \tag{4.62}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+\alpha) \\ (e_6|b_3+\alpha) \\ (z_1|b_3+\alpha) \\ -1/2+(\alpha+z_2|b_3+\alpha) \end{pmatrix} \tag{4.63}$$

Sectors $B_{pqrs}^{(16-)}$, $B_{pqrs}^{(17-)}$, $B_{pqrs}^{(18-)}$

$$B_{pqrs}^{(16-,17-,18-)} = B_{pqrs}^{(1,2,3)} - \alpha \tag{4.64}$$

States $\{\bar{\eta}^{1*}\}|R>$ and $\{\bar{\psi}^{1..5\,*}\}|R>$

$$\begin{aligned} P_{pqrs}^{((15+i)-)(\bar{\eta}^{1*};\bar{\psi}^{1..5\,*})} = \tfrac{1}{16} & \left(1 - C\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \\ & \cdot \left(1 + C\begin{pmatrix} z_1 \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \cdot \left(1 + i\,C\begin{pmatrix} \alpha + z_2 \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \end{aligned} \tag{4.65}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1-\alpha) \\ (e_2|b_1-\alpha) \\ 1+(z_1|b_1-\alpha) \\ -1/2+(\alpha+z_2|b_1-\alpha) \end{pmatrix} \tag{4.66}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2-\alpha) \\ (e_4|b_2-\alpha) \\ 1+(z_1|b_2-\alpha) \\ -1/2+(\alpha+z_2|b_2-\alpha) \end{pmatrix} \tag{4.67}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3-\alpha) \\ (e_6|b_3-\alpha) \\ 1+(z_1|b_3-\alpha) \\ -1/2+(\alpha+z_2|b_3-\alpha) \end{pmatrix} \tag{4.68}$$

States $\{\bar{\eta}^2\}|R>$ and $\{\bar{\eta}^3\}|R>$

$$\begin{aligned} P_{pqrs}^{((15+i)-)(\bar{\eta}^2;\bar{\eta}^3)} = \tfrac{1}{16} & \left(1 - C\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \\ & \cdot \left(1 + C\begin{pmatrix} z_1 \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \cdot \left(1 - i\,C\begin{pmatrix} \alpha + z_2 \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \end{aligned} \tag{4.69}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1-\alpha) \\ (e_2|b_1-\alpha) \\ 1+(z_1|b_1-\alpha) \\ 1/2+(\alpha+z_2|b_1-\alpha) \end{pmatrix} \tag{4.70}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2-\alpha) \\ (e_4|b_2-\alpha) \\ 1+(z_1|b_2-\alpha) \\ 1/2+(\alpha+z_2|b_2-\alpha) \end{pmatrix} \tag{4.71}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3-\alpha) \\ (e_6|b_3-\alpha) \\ 1+(z_1|b_3-\alpha) \\ 1/2+(\alpha+z_2|b_3-\alpha) \end{pmatrix} \tag{4.72}$$

States $\{\bar{\varphi}^{1..4}\}|R>$

$$\begin{aligned} P_{pqrs}^{((15+i)-)(\bar{\varphi}^{1..4})} = \tfrac{1}{16} & \left(1-C\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \cdot \left(1-C\begin{pmatrix} e_{2i} \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \\ & \cdot \left(1-C\begin{pmatrix} z_1 \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \cdot \left(1-i\,C\begin{pmatrix} \alpha+z_2 \\ B_{pqrs}^{((15+i)-)} \end{pmatrix}\right) \end{aligned} \tag{4.73}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1-\alpha) \\ (e_2|b_1-\alpha) \\ (z_1|b_1-\alpha) \\ 1/2+(\alpha+z_2|b_1-\alpha) \end{pmatrix} \tag{4.74}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2-\alpha) \\ (e_4|b_2-\alpha) \\ (z_1|b_2-\alpha) \\ 1/2+(\alpha+z_2|b_2-\alpha) \end{pmatrix} \tag{4.75}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3-\alpha) \\ (e_6|b_3-\alpha) \\ (z_1|b_3-\alpha) \\ 1/2+(\alpha+z_2|b_3-\alpha) \end{pmatrix} \tag{4.76}$$

Sectors $B_{pqrs}^{(19+)}$, $B_{pqrs}^{(20+)}$, $B_{pqrs}^{(21+)}$

$$B_{pqrs}^{(19+,20+,21+)} = B_{pqrs}^{(1,2,3)} + \alpha + z_1 \tag{4.77}$$

States $\{\bar{\eta}^1\}|R>$ and $\{\bar{\psi}^{1..5}\}|R>$

$$\begin{aligned} P_{pqrs}^{((18+i)+)(\bar{\eta}^1;\bar{\psi}^{1..5})} = \tfrac{1}{16} \quad & \left(1 - C\binom{e_{2i-1}}{B_{pqrs}^{((18+i)+)}}\right) \cdot \left(1 - C\binom{e_{2i}}{B_{pqrs}^{((18+i)+)}}\right) \\ & \cdot \left(1 + C\binom{z_1}{B_{pqrs}^{((18+i)+)}}\right) \cdot \left(1 + i\,C\binom{\alpha + z_2}{B_{pqrs}^{((18+i)+)}}\right) \end{aligned} \tag{4.78}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha+z_1) \\ (e_2|b_1+\alpha+z_1) \\ 1+(z_1|b_1+\alpha+z_1) \\ -1/2+(\alpha+z_2|b_1+\alpha+z_1) \end{pmatrix} \tag{4.79}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+\alpha+z_1) \\ (e_4|b_2+\alpha+z_1) \\ 1+(z_1|b_2+\alpha+z_1) \\ -1/2+(\alpha+z_2|b_2+\alpha+z_1) \end{pmatrix} \tag{4.80}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+\alpha+z_1) \\ (e_6|b_3+\alpha+z_1) \\ 1+(z_1|b_3+\alpha+z_1) \\ -1/2+(\alpha+z_2|b_3+\alpha+z_1) \end{pmatrix} \tag{4.81}$$

States $\{\bar{\eta}^{2*}\}|R>$ and $\{\bar{\eta}^{3*}\}|R>$

$$\begin{aligned} P_{pqrs}^{((18+i)+)(\bar{\eta}^{2*};\bar{\eta}^{3*})} = \tfrac{1}{16} \quad & \left(1 - C\binom{e_{2i-1}}{B_{pqrs}^{((18+i)+)}}\right) \cdot \left(1 - C\binom{e_{2i}}{B_{pqrs}^{((18+i)+)}}\right) \\ & \cdot \left(1 + C\binom{z_1}{B_{pqrs}^{((18+i)+)}}\right) \cdot \left(1 - i\,C\binom{\alpha + z_2}{B_{pqrs}^{((18+i)+)}}\right) \end{aligned} \tag{4.82}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha+z_1) \\ (e_2|b_1+\alpha+z_1) \\ 1+(z_1|b_1+\alpha+z_1) \\ 1/2+(\alpha+z_2|b_1+\alpha+z_1) \end{pmatrix} \tag{4.83}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+\alpha+z_1) \\ (e_4|b_2+\alpha+z_1) \\ 1+(z_1|b_2+\alpha+z_1) \\ 1/2+(\alpha+z_2|b_2+\alpha+z_1) \end{pmatrix} \tag{4.84}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+\alpha+z_1) \\ (e_6|b_3+\alpha+z_1) \\ 1+(z_1|b_3+\alpha+z_1) \\ 1/2+(\alpha+z_2|b_3+\alpha+z_1) \end{pmatrix} \tag{4.85}$$

States $\{\bar{\varphi}^{1..4}\}|R>$

$$\begin{aligned} P_{pqrs}^{((18+i)+)(\bar{\varphi}^{1..4*})} = \tfrac{1}{16} \quad & \left(1-C\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{((18+i)+)} \end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_{2i} \\ B_{pqrs}^{((18+i)+)} \end{pmatrix}\right) \\ & \cdot\left(1-C\begin{pmatrix} z_1 \\ B_{pqrs}^{((18+i)+)} \end{pmatrix}\right)\cdot\left(1+i\,C\begin{pmatrix} \alpha+z_2 \\ B_{pqrs}^{((18+i)+)} \end{pmatrix}\right) \end{aligned} \tag{4.86}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha+z_1) \\ (e_2|b_1+\alpha+z_1) \\ (z_1|b_1+\alpha+z_1) \\ -1/2+(\alpha+z_2|b_1+\alpha+z_1) \end{pmatrix} \tag{4.87}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+\alpha+z_1) \\ (e_4|b_2+\alpha+z_1) \\ (z_1|b_2+\alpha+z_1) \\ -1/2+(\alpha+z_2|b_2+\alpha+z_1) \end{pmatrix} \tag{4.88}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+\alpha+z_1) \\ (e_6|b_3+\alpha+z_1) \\ (z_1|b_3+\alpha+z_1) \\ -1/2+(\alpha+z_2|b_3+\alpha+z_1) \end{pmatrix} \tag{4.89}$$

Sectors $B_{pqrs}^{(19-)}$, $B_{pqrs}^{(20-)}$, $B_{pqrs}^{(21-)}$

$$B_{pqrs}^{(19-,20-,21-)} = B_{pqrs}^{(1,2,3)} - \alpha + z_1 \tag{4.90}$$

States $\{\bar{\eta}^{1*}\}|R>$ and $\{\bar{\psi}^{1..5\,*}\}|R>$

$$\begin{aligned} P_{pqrs}^{((18+i)-)(\bar{\eta}^{1*};\bar{\psi}^{1..5\,*})} = \tfrac{1}{16} & \left(1 - C\binom{e_{2i-1}}{B_{pqrs}^{((18+i)-)}}\right) \cdot \left(1 - C\binom{e_{2i}}{B_{pqrs}^{((18+i)-)}}\right) \\ & \cdot \left(1 + C\binom{z_1}{B_{pqrs}^{((18+i)-)}}\right) \cdot \left(1 - i\,C\binom{\alpha + z_2}{B_{pqrs}^{((18+i)-)}}\right) \end{aligned} \tag{4.91}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 - \alpha + z_1) \\ (e_2|b_1 - \alpha + z_1) \\ 1 + (z_1|b_1 - \alpha + z_1) \\ 1/2 + (\alpha + z_2|b_1 - \alpha + z_1) \end{pmatrix} \tag{4.92}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 - \alpha + z_1) \\ (e_4|b_2 - \alpha + z_1) \\ 1 + (z_1|b_2 - \alpha + z_1) \\ 1/2 + (\alpha + z_2|b_2 - \alpha + z_1) \end{pmatrix} \tag{4.93}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3 - \alpha + z_1) \\ (e_6|b_3 - \alpha + z_1) \\ 1 + (z_1|b_3 - \alpha + z_1) \\ 1/2 + (\alpha + z_2|b_3 - \alpha + z_1) \end{pmatrix} \tag{4.94}$$

States $\{\bar{\eta}^{2}\}|R>$ and $\{\bar{\eta}^{3}\}|R>$

$$\begin{aligned} P_{pqrs}^{((18+i)-)(\bar{\eta}^{2};\bar{\eta}^{3})} = \tfrac{1}{16} & \left(1 - C\binom{e_{2i-1}}{B_{pqrs}^{((18+i)-)}}\right) \cdot \left(1 - C\binom{e_{2i}}{B_{pqrs}^{((18+i)-)}}\right) \\ & \cdot \left(1 + C\binom{z_1}{B_{pqrs}^{((18+i)-)}}\right) \cdot \left(1 + i\,C\binom{\alpha + z_2}{B_{pqrs}^{((18+i)-)}}\right) \end{aligned} \tag{4.95}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1-\alpha+z_1) \\ (e_2|b_1-\alpha+z_1) \\ 1+(z_1|b_1-\alpha+z_1) \\ -1/2+(\alpha+z_2|b_1-\alpha+z_1) \end{pmatrix} \quad (4.96)$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2-\alpha+z_1) \\ (e_4|b_2-\alpha+z_1) \\ 1+(z_1|b_2-\alpha+z_1) \\ -1/2+(\alpha+z_2|b_2-\alpha+z_1) \end{pmatrix} \quad (4.97)$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3-\alpha+z_1) \\ (e_6|b_3-\alpha+z_1) \\ 1+(z_1|b_3-\alpha+z_1) \\ -1/2+(\alpha+z_2|b_3-\alpha+z_1) \end{pmatrix} \quad (4.98)$$

States $\{\bar{\varphi}^{1..4\,*}\}|R>$

$$P_{pqrs}^{((18+i)-)(\bar{\varphi}^{1..4\,*})} = \frac{1}{16} \left(1 - C\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{((18+i)-)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B_{pqrs}^{((18+i)-)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} z_1 \\ B_{pqrs}^{((18+i)-)} \end{pmatrix}\right) \cdot \left(1 - i\,C\begin{pmatrix} \alpha + z_2 \\ B_{pqrs}^{((18+i)-)} \end{pmatrix}\right) \quad (4.99)$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_3) & (z_2+\alpha|e_4) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1-\alpha+z_1) \\ (e_2|b_1-\alpha+z_1) \\ (z_1|b_1-\alpha+z_1) \\ 1/2+(\alpha+z_2|b_1-\alpha+z_1) \end{pmatrix} \quad (4.100)$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_5) & (z_2+\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2-\alpha+z_1) \\ (e_4|b_2-\alpha+z_1) \\ (z_1|b_2-\alpha+z_1) \\ 1/2+(\alpha+z_2|b_2-\alpha+z_1) \end{pmatrix} \quad (4.101)$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2+\alpha|e_1) & (z_2+\alpha|e_2) & (z_2+\alpha|e_3) & (z_2+\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3-\alpha+z_1) \\ (e_6|b_3-\alpha+z_1) \\ (z_1|b_3-\alpha+z_1) \\ 1/2+(\alpha+z_2|b_3-\alpha+z_1) \end{pmatrix} \quad (4.102)$$

Sectors $B^{(22)}_{pqrs}$, $B^{(23)}_{pqrs}$, $B^{(24)}_{pqrs}$

$$B^{(22,23,24)}_{pqrs} = B^{(1,2,3)}_{pqrs} + 2\alpha + z_1 \tag{4.103}$$

States $\{\bar{\eta}^1\}|R>$ and $\{\bar{\psi}^{1..5}\}|R>$ and $\{\bar{\eta}^{1*}\}|R>$ and $\{\bar{\psi}^{1..5*}\}|R>$

$$\begin{aligned} P^{(21+i)(\bar{\eta}^1;\bar{\psi}^{1..5})}_{pqrs} = \frac{1}{16} & \left(1 - C\begin{pmatrix} e_{2i-1} \\ B^{(21+i)}_{pqrs}\end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B^{(21+i)}_{pqrs}\end{pmatrix}\right) \\ & \cdot \left(1 - C\begin{pmatrix} z_1 \\ B^{(21+i)}_{pqrs}\end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} z_2 \\ B^{(21+i)}_{pqrs}\end{pmatrix}\right) \end{aligned} \tag{4.104}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + 2\alpha + z_1) \\ (e_2|b_1 + 2\alpha + z_1) \\ (z_1|b_1 + 2\alpha + z_1) \\ (z_2|b_1 + 2\alpha + z_1) \end{pmatrix} \tag{4.105}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + 2\alpha + z_1) \\ (e_4|b_2 + 2\alpha + z_1) \\ (z_1|b_2 + 2\alpha + z_1) \\ (z_2|b_2 + 2\alpha + z_1) \end{pmatrix} \tag{4.106}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3 + 2\alpha + z_1) \\ (e_6|b_3 + 2\alpha + z_1) \\ (z_1|b_3 + 2\alpha + z_1) \\ (z_2|b_3 + 2\alpha + z_1) \end{pmatrix} \tag{4.107}$$

States $\{\bar{\varphi}^{1..4}\}|R>$ and $\{\bar{\varphi}^{1..4*}\}|R>$

$$\begin{aligned} P^{(21+i)(\bar{\varphi}^{1..4})}_{pqrs} = \frac{1}{16} & \left(1 - C\begin{pmatrix} e_{2i-1} \\ B^{(21+i)}_{pqrs}\end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B^{(21+i)}_{pqrs}\end{pmatrix}\right) \\ & \cdot \left(1 + C\begin{pmatrix} z_1 \\ B^{(21+i)}_{pqrs}\end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} z_2 \\ B^{(21+i)}_{pqrs}\end{pmatrix}\right) \end{aligned} \tag{4.108}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + 2\alpha + z_1) \\ (e_2|b_1 + 2\alpha + z_1) \\ 1 + (z_1|b_1 + 2\alpha + z_1) \\ (z_2|b_1 + 2\alpha + z_1) \end{pmatrix} \tag{4.109}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + 2\alpha + z_1) \\ (e_4|b_2 + 2\alpha + z_1) \\ 1 + (z_1|b_2 + 2\alpha + z_1) \\ (z_2|b_2 + 2\alpha + z_1) \end{pmatrix} \tag{4.110}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3 + 2\alpha + z_1) \\ (e_6|b_3 + 2\alpha + z_1) \\ 1 + (z_1|b_3 + 2\alpha + z_1) \\ (z_2|b_3 + 2\alpha + z_1) \end{pmatrix} \tag{4.111}$$

States $\{\bar{\varphi}^{6..8}\}|R>$, $\{\bar{\varphi}^{6..8\,*}\}|R>$, $\{\bar{\varphi}^{5}\}|R>$ and $\{\bar{\varphi}^{5\,*}\}|R>$

$$\begin{aligned} P_{pqrs}^{(21+i)(\bar{\varphi}^5;\bar{\varphi}^{6..8})} = \frac{1}{16} & \left(1 - C\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{(21+i)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_{2i} \\ B_{pqrs}^{(21+i)} \end{pmatrix}\right) \\ & \cdot \left(1 - C\begin{pmatrix} z_1 \\ B_{pqrs}^{(21+i)} \end{pmatrix}\right) \cdot \left(1 + C\begin{pmatrix} z_2 \\ B_{pqrs}^{(21+i)} \end{pmatrix}\right) \end{aligned} \tag{4.112}$$

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + 2\alpha + z_1) \\ (e_2|b_1 + 2\alpha + z_1) \\ (z_1|b_1 + 2\alpha + z_1) \\ 1 + (z_2|b_1 + 2\alpha + z_1) \end{pmatrix} \tag{4.113}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + 2\alpha + z_1) \\ (e_4|b_2 + 2\alpha + z_1) \\ (z_1|b_2 + 2\alpha + z_1) \\ 1 + (z_2|b_2 + 2\alpha + z_1) \end{pmatrix} \tag{4.114}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3 + 2\alpha + z_1) \\ (e_6|b_3 + 2\alpha + z_1) \\ (z_1|b_3 + 2\alpha + z_1) \\ 1 + (z_2|b_3 + 2\alpha + z_1) \end{pmatrix} \tag{4.115}$$

B. F-flatness constraints

$$\mathbf{F_1} \;:\; F_2\,F_4 + \bar{f}_3\,f_4 + 2\,F_1\,F_5 + \bar{F}_5\,\chi_1 + \frac{1}{\sqrt{2}}\,\bar{F}_4\,\bar{\xi}_{11} = 0\;, \tag{4.116}$$

$$\mathbf{F_2} \;:\; F_1\,F_4 + 2\,F_2\,F_6 + \bar{F}_6\,\chi_2 = 0\;, \tag{4.117}$$

$$\mathbf{F_3} \;:\; \bar{f}_4\,f_1 + 2\,F_3\,F_6 + \bar{F}_6\,\chi_3 = 0\;, \tag{4.118}$$

$$\mathbf{F_4} \;:\; F_1\,F_2 + \bar{f}_1\,\bar{f}_2 + 2\,F_4\,F_7 + \bar{F}_7\,\chi_4 = 0\;, \tag{4.119}$$

$$\mathbf{\bar{F}_4} \;:\; 2\,\bar{F}_4\,\bar{F}_7 + F_7\,\chi_5 + \frac{1}{\sqrt{2}}\,F_1\,\bar{\xi}_{11} = 0\;, \tag{4.120}$$

$$\mathbf{\bar{f}_1} \;:\; \bar{f}_2\,F_4 + \bar{f}_4\,F_3 + 2\,\bar{f}_1\,F_5 + \frac{1}{\sqrt{2}}\,f_1\,\bar{\xi}_{12} = 0\;, \tag{4.121}$$

$$\mathbf{\bar{f}_2} \;:\; \bar{f}_1\,F_4 + 2\,\bar{f}_2\,F_6 = 0\;, \tag{4.122}$$

$$\mathbf{\bar{f}_3} \;:\; \bar{f}_4\,F_1 + 2\,\bar{f}_3\,F_6 = 0\;, \tag{4.123}$$

$$\mathbf{f_1} \;:\; 2\,f_1\,\bar{F}_7 + \frac{1}{\sqrt{2}}\,\bar{f}_1\,\bar{\xi}_{12} = 0\;, \tag{4.124}$$

$$\mathbf{\bar{f}_4} \;:\; \bar{f}_3\,F_1 + \bar{f}_1\,F_3 + 2\,\bar{f}_4\,F_7 = 0\;, \tag{4.125}$$

$$\mathbf{F_5} \;:\; {F_1}^2 + {\bar{f}_1}^{\,2} + F_6\,F_7 + \bar{F}_6\,\Phi_{56} + \bar{F}_7\,\Phi_{34} = 0\;, \tag{4.126}$$

$$\mathbf{F_6} \;:\; {F_2}^2 + {\bar{f}_2}^{\,2} + {F_3}^2 + {\bar{f}_3}^{\,2} + F_5\,F_7 + \bar{F}_5\,\bar{\Phi}_{56} + \bar{F}_7\,\Phi_{12} = 0\;, \tag{4.127}$$

$$\mathbf{F_7} \;:\; {F_4}^2 + {\bar{f}_4}^{\,2} + F_4\,\chi_5 + F_5\,F_6 + \bar{F}_5\,\bar{\Phi}_{34} + \bar{F}_6\,\bar{\Phi}_{12} = 0\;, \tag{4.128}$$

$$\mathbf{\bar{F}_5} \;:\; F_1\,\chi_1 + \bar{F}_6\,\bar{F}_7 + F_6\,\bar{\Phi}_{56} + F_7\,\bar{\Phi}_{34} = 0\;, \tag{4.129}$$

$$\mathbf{\bar{F}_6} \;:\; F_2\,\chi_2 + F_3\,\chi_3 + \bar{F}_5\,\bar{F}_7 + F_5\,\Phi_{56} + F_7\,\bar{\Phi}_{12} = 0\;, \tag{4.130}$$

$$\mathbf{\bar{F}_7} \;:\; {f_1}^2 + {\bar{F}_4}^{\,2} + F_4\,\chi_4 + \bar{F}_5\,\bar{F}_6 + F_5\,\Phi_{34} + F_6\,\Phi_{12} = 0\;, \tag{4.131}$$

$$\mathbf{\Phi_{12}} \;:\; \bar{\Phi}_{34}\,\Phi_{56} + \bar{F}_7\,F_6 + {\bar{\xi}_1}^{\,2} + {\bar{\xi}_2}^{\,2} = 0\;, \tag{4.132}$$

$$\mathbf{\Phi_{34}} \;:\; \bar{\Phi}_{12}\,\bar{\Phi}_{56} + \bar{F}_7\,F_5 + {\bar{\xi}_3}^{\,2} + {\bar{\xi}_4}^{\,2} + {\bar{\xi}_5}^{\,2} + {\bar{\xi}_6}^{\,2} + \bar{\xi}_{11}^{\,2} + \bar{\xi}_{12}^{\,2} = 0\;, \tag{4.133}$$

$$\mathbf{\Phi_{56}} \;:\; \Phi_{12}\,\bar{\Phi}_{34} + \bar{F}_6\,F_5 + {\bar{\xi}_7}^{\,2} + {\bar{\xi}_8}^{\,2} + {\xi_9}^2 + \xi_{10}^2 = 0\;, \tag{4.134}$$

$$\mathbf{\bar{\Phi}_{12}} \;:\; \Phi_{34}\,\bar{\Phi}_{56} + \bar{F}_6\,F_7 + {\xi_1}^2 + {\xi_2}^2 = 0\;, \tag{4.135}$$

$$\mathbf{\bar{\Phi}_{34}} \;:\; \Phi_{12}\,\Phi_{56} + \bar{F}_5\,F_7 + {\xi_3}^2 + {\xi_4}^2 + {\xi_5}^2 + {\xi_6}^2 + \xi_{11}^2 + \xi_{12}^2 = 0\;, \tag{4.136}$$

$$\mathbf{\bar{\Phi}_{56}} \;:\; \Phi_{34}\,\bar{\Phi}_{12} + \bar{F}_5\,F_6 + {\xi_7}^2 + {\xi_8}^2 + {\bar{\xi}_9}^{\,2} + \bar{\xi}_{10}^{\,2} = 0\;, \tag{4.137}$$

$$\mathbf{\Phi_4} \;:\; \xi_{11}\,\bar{\xi}_{11} + \xi_{12}\,\bar{\xi}_{12} = 0\;, \tag{4.138}$$

$$\boldsymbol{\chi_1} \;:\; \bar{F}_5\,F_1 + \chi_2\,\chi_4 + \xi_3\,\xi_7 + \xi_4\,\xi_8 + \bar{\xi}_{10}\,\xi_{11} = 0\;, \tag{4.139}$$

$$\boldsymbol{\chi_2} \;:\; \bar{F}_6\,F_2 + \chi_1\,\chi_4 + \xi_2\,\bar{\xi}_7 + \xi_1\,\bar{\xi}_8 = 0\;, \tag{4.140}$$

$$\boldsymbol{\chi_4} \;:\; \bar{F}_7\,F_4 + \chi_1\,\chi_2 + \bar{\xi}_1\,\bar{\xi}_3 + \bar{\xi}_2\,\bar{\xi}_4 = 0\;, \tag{4.141}$$

$$\boldsymbol{\chi_3} \;:\; \bar{F}_6\,F_3 = 0\;, \tag{4.142}$$

$$\boldsymbol{\chi_5} \;:\; F_7\,\bar{F}_4 + \xi_1\,\xi_{11} = 0\;, \tag{4.143}$$

$$\xi_{\mathbf{4}} \;:\; \xi_8\,\chi_1 + 2\,\xi_4\,\bar\Phi_{56} = 0\ , \tag{4.144}$$
$$\xi_{\mathbf{6}} \;:\; 2\,\xi_6\,\bar\Phi_{34} = 0\ , \tag{4.145}$$
$$\xi_{\mathbf{3}} \;:\; \xi_7\,\chi_1 + 2\,\xi_3\,\bar\Phi_{56} = 0\ , \tag{4.146}$$
$$\xi_{\mathbf{10}} \;:\; 2\,\xi_{10}\,\Phi_{56} = 0\ , \tag{4.147}$$

$$\xi_{\mathbf{7}} \;:\; \xi_3\,\chi_1 + 2\,\xi_7\,\bar\Phi_{56} = 0\ , \tag{4.148}$$
$$\xi_{\mathbf{5}} \;:\; 2\,\xi_5\,\bar\Phi_{34} = 0\ , \tag{4.149}$$
$$\bar\xi_{\mathbf{10}} \;:\; \xi_{11}\,\chi_1 + 2\,\bar\xi_{10}\,\bar\Phi_{56} + \frac{1}{\sqrt{2}}\,\xi_1\,\bar\xi_{11} = 0\ , \tag{4.150}$$
$$\xi_{\mathbf{9}} \;:\; 2\,\xi_9\,\Phi_{56} = 0\ , \tag{4.151}$$

$$\xi_{\mathbf{2}} \;:\; \bar\xi_7\,\chi_2 + 2\,\xi_2\,\bar\Phi_{12} + \frac{1}{\sqrt{2}}\,\bar\xi_9\,\bar\xi_{11} = 0\ , \tag{4.152}$$
$$\bar\xi_{\mathbf{8}} \;:\; \xi_1\,\chi_2 + 2\,\bar\xi_8\,\Phi_{56} = 0\ , \tag{4.153}$$
$$\bar\xi_{\mathbf{9}} \;:\; 2\,\bar\xi_9\,\bar\Phi_{56} + \frac{1}{\sqrt{2}}\,\xi_2\,\bar\xi_{11} = 0\ , \tag{4.154}$$

$$\xi_{\mathbf{1}} \;:\; \bar\xi_8\,\chi_2 + \xi_{11}\,\chi_5 + 2\,\xi_1\,\bar\Phi_{12} + \frac{1}{\sqrt{2}}\,\bar\xi_{10}\,\bar\xi_{11} = 0\ , \tag{4.155}$$
$$\bar\xi_{\mathbf{7}} \;:\; \xi_2\,\chi_2 + 2\,\bar\xi_7\,\Phi_{56} = 0\ , \tag{4.156}$$
$$\xi_{\mathbf{8}} \;:\; \xi_4\,\chi_1 + 2\,\xi_8\,\bar\Phi_{56} = 0\ , \tag{4.157}$$
$$\xi_{\mathbf{11}} \;:\; \bar\xi_{10}\,\chi_1 + \xi_1\,\chi_5 + 2\,\xi_{11}\,\bar\Phi_{34} + \bar\xi_{11}\,\Phi_4 = 0\ , \tag{4.158}$$
$$\bar\xi_{\mathbf{11}} \;:\; 2\,\bar\xi_{11}\,\Phi_{34} + \xi_{11}\,\Phi_4 + \frac{1}{\sqrt{2}}\left\{\,F_1\,\bar F_4 + \xi_1\,\bar\xi_{10} + \xi_2\,\bar\xi_9\,\right\} = 0\ , \tag{4.159}$$
$$\xi_{\mathbf{12}} \;:\; 2\,\xi_{12}\,\bar\Phi_{34} + \bar\xi_{12}\,\Phi_4 = 0\ , \tag{4.160}$$
$$\bar\xi_{\mathbf{12}} \;:\; 2\,\bar\xi_{12}\,\Phi_{34} + \xi_{12}\,\Phi_4 + \frac{1}{\sqrt{2}}\,f_1\,\bar f_1 = 0\ , \tag{4.161}$$

$$\bar\xi_{\mathbf{1}} \;:\; \bar\xi_3\,\chi_4 + 2\,\bar\xi_1\,\Phi_{12} = 0\ , \tag{4.162}$$
$$\bar\xi_{\mathbf{3}} \;:\; \bar\xi_1\,\chi_4 + 2\,\bar\xi_5\,\Phi_{34} = 0\ , \tag{4.163}$$
$$\bar\xi_{\mathbf{5}} \;:\; 2\,\bar\xi_5\,\Phi_{34} = 0\ , \tag{4.164}$$

$$\bar\xi_{\mathbf{2}} \;:\; \bar\xi_4\,\chi_4 + 2\,\bar\xi_2\,\Phi_{12} = 0\ , \tag{4.165}$$
$$\bar\xi_{\mathbf{4}} \;:\; \bar\xi_2\,\chi_4 + 2\,\bar\xi_4\,\Phi_{34} = 0\ , \tag{4.166}$$
$$\bar\xi_{\mathbf{6}} \;:\; 2\,\bar\xi_6\,\Phi_{34} = 0\ . \tag{4.167}$$

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