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## Classification of free fermionic models in 4D-heterotic string theory

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**Abstract**

In this report we detail a classification of a group of free fermionic models in four-dimensionnal heterotic theory. This classification is basically going a step further in the classification undertaken in paper [11]. We mainly expose calculations that will be used to do computationnal work and then provide an insight in the properties of this class of models in terms of matter content. So this report should be regarded as uncomplete, since it deals more about methodology than results.
In part two we expose briefly the basis of string theory, from classical strings to superstrings, then in part three we move to heterotic strings and more specifically free fermionic models, which is our frame for the classification. Finally we detail the methodology and calculations for studying the gauge group and matter content of $SO(6) \times SO(4)$ models and we expose our expectations and ideas for future work.

Dans ce rapport nous présentons une classification d'un groupe de modèles de type libres fermions dans le cadre d'une theorie des cordes hétérotiques à quatre dimensions. Cette classification poursuit le travail entrepris dans l'article [11]. Nous exposons les calculs qui seront utilisés dans un travail statistique informatique à venir et présentons le contenu de cette classe de modèles en termes de particules. Ce rapport n'est donc que la première partie du projet de recherche, puisqu'il traite principalement de méthodologie et prépare des résultats futurs.
Dans la seconde partie nous exposons rapidement les bases de la théorie des cordes, depuis la corde classique jusqu'aux supercordes, puis, dans la partie trois nous introduisons les cordes hétérotiques et plus particulièrement les modèles de type fermions libres, qui constituent le cadre de notre étude. Enfin nous détaillons la méthodologie et les calculs pour étudier le groupe de gauge et le contenu physique des modèles $SO(6) \times SO(4)$ et nous exposons nos attentes ainsi que quelques idées pour poursuivre cette étude.

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# Contents

# 1 Introduction

During these three months I have studied a specific branch of string theory - free fermionic heterotic models - and contributed to do some calculations providing a description of a class of models one can obtain from ABK rules [1, 2] (Antoniadis, Bachas, Kounnas).
String theory no longer considers the fundamental objects in physics to be point particules but one dimensional objects called stings or more generally D-dimensional objects called D-branes. The main achievement of this theory is to provide a frame for a possible quantum description of gravity. This makes string theory a good candidate for a grand unified theory of physics (GUT). For a few decades a lot of efforts have been put in trying to see the Standard Model (SM) or the Minimal Supersymetric Standard Model (MSSM) emerge from the matter content of string theory, altogether with some acknowledged properties like the stability of the proton (half life $> 10^{32}$ years).
The ABK rules provide a frame to develop four dimensional models in the particular context of heterotic strings, as will be explained below. The ultimate purpose of these constructions is to obtain an explicit expression for the partition function of a one-loop diagram and then to derive predictions perturbatively.
The difficulty in this procedure is to understand how to select the relevant models in the infinity of possible models. A first step in this research consists in looking at the content in terms of gauge bosons and chiral families of those models in order to know which of them could contain the Standard Model (or MSSM).
The main part of my project has been to describe the gauge groups and the matter contents of one class of models and to set some algorithmic rules to derive them from the free parameters of the theory. A similar work had already been completed for a simplier class of model, so what I have done is basically to go a step further in this classification.
The last stage of this project is to compute the calculations and try to see some properties emerging concerning the number of chiral families. This computationnal work will be achieve in the next months by another person involved in the project.

# 2 Strings and superstrings

One can find an introduction to string theory and superstrings in references [4, 5, 6].
The first step in building string theory is the quantization of the classical string, providing a theory of free bosonic strings.

## 2.1 Classical strings

We consider a fixed Pseudo-Riemannian space-time $M$ of dimension D, with coordinates $X = (X^\mu), \mu = 0, 1, ..., D-1$ and a metric $G_{\mu\nu}(X)$.
The motion of a relativistic string in $M$ is described by a two dimensional surface $\Sigma$, which is called the world-sheet. We introduce coordinates $\sigma = (\sigma^0, \sigma^1)$ on the world-sheet, so that we have the map :

$$X : \Sigma \longrightarrow M : \sigma \longrightarrow X(\sigma) \tag{1}$$

We take $\sigma^0$ to be the time-like coordinate (taking its values from $-\infty$ to $+\infty$) and $\sigma^1$ to be space-like (taking its values in a finite interval) and parameterizing the string at any given instant.

The natural action for the relativistic string is its area (Nambu-Goto action):

$$S_{NG} = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma |det\, G_{\alpha\beta}|^{1/2}, \tag{2}$$

where $\frac{1}{2\pi\alpha'}$ is the energy per length (or tension) of the string and $G_{\alpha\beta}$ is the induced metric on the world-sheet :

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} G_{\mu\nu}, \tag{3}$$

$\alpha, \beta = 0, 1$ are world-sheet indices.
An equivalent and more convenient expression of the action is given by the Polyakov action, which reduces in a standard gauge, called conformal gauge, to :

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma\, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X), \tag{4}$$

with $\eta_{\alpha\beta} = \text{Diag}(-1, 1)$.

We now specify the metric to be the Minkovsky metric $G_{\mu\nu}(X) = \eta_{\mu\nu} = \text{Diag}(-1, 1, ..., 1)$.
The equation of motion then reduces to a free wave equation :

$$\partial^2 X^\mu(\sigma) = \partial^\alpha \partial_\alpha X^\mu(\sigma) = 0. \tag{5}$$

The general solution of this equation is a superposition of left- and right-moving waves :

$$X^\mu(\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-), \tag{6}$$

where $\sigma^+ = \sigma^0 + \sigma^1$ and $\sigma^- = \sigma^0 - \sigma^1$.

To go further we have to specify the boundary conditions at the end of the string.
One possible choice is to take **periodic boundary conditions**, which correspond to closed strings :

$$X^\mu(\sigma^0, \sigma^1 + \pi) = X^\mu(\sigma^0, \sigma^1), \tag{7}$$

where we assumed that $\sigma^1 \in [0, \pi]$ parameterizes the full string for a given $\sigma^0$.
In this case we can express the solutions of the wave equation :

$$X^\mu(\sigma) = x^\mu + 2\alpha' p^\mu \sigma^0 + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{\alpha_n^\mu}{n} e^{-2in\sigma^+} + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-2in\sigma^-}, \tag{8}$$

where $x^\mu, p^\mu$ are real parameters, $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$ and $(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu$ (due to the reality of $X^\mu$). $x^\mu$ is the position of the center of mass of the string and $p^\mu$ is its total momentum.

If we do not take the periodic boundary conditions, then we need to impose another set of constraints to make the variation of the action vanish and preserve the wave equation. There are two possible choices :

- the Dirichlet boundary conditions,

$$X^\mu(\sigma^1 = 0) = \text{constant}_1, \quad X^\mu(\sigma^1 = \pi) = \text{constant}_2, \tag{9}$$

- the Neumann boundary conditions,

$$\partial_1 X^\mu(\sigma^1 = 0) = 0, \quad \partial_1 X^\mu(\sigma^1 = \pi) = 0. \tag{10}$$

These conditions lead to expressions of $X^\mu(\sigma)$ in terms of oscillators as in the closed string case, but now the left- and right-moving waves are coupled so that we end up with a superposition of stationary waves. These solutions describe what is called the open string.
Since in the rest of this report we will only work with closed strings, I will not develop further the formalism of open strings.

## 2.2 Quantized bosonic strings

We now state that the parameters $x^\mu$, $p^\mu$, $\alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$ become quantum operators satifying the following canonical commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \tag{11}$$
$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\eta^{\mu\nu}\delta_{m+n,0}, \tag{12}$$
$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\,\eta^{\mu\nu}\delta_{m+n,0}. \tag{13}$$

and other commutators equal zero.
These canonical relations can be deduced from the required conditions

$$[X^\mu(\sigma^0,\sigma^1), \frac{1}{2\pi\alpha'}\partial_0 X^\nu(\sigma^0,\sigma^{1'})] = i\eta^{\mu\nu}\delta(\sigma^1-\sigma^{1'}), \tag{14}$$

which are the commutation relations of the canonical quantization procedure.
The reality conditions become hermiticity relations $(x^\mu)^+ = x^\mu$, $(p^\mu)^+ = p^\mu$, $(\alpha_n^\mu)^+ = \alpha_{-n}^\mu$ and $(\tilde{\alpha}_n^\mu)^+ = \tilde{\alpha}_{-n}^\mu$.
One can introduce the operators

$$a_n^\mu = \frac{\alpha_n^\mu}{\sqrt{|n|}}, \quad \tilde{a}_n^\mu = \frac{\tilde{\alpha}_n^\mu}{\sqrt{|n|}}, \quad n \neq 0, \tag{15}$$

which satisfy the commutation relations of creation ($n > 0$) and annihilation ($n < 0$) operators.

Let us now remark that in this construction we ignore the fact that we are free to impose some conditions the world-sheet coordinate $\sigma^0, \sigma^1$. Imposing these conditions is fixing a gauge. A particularly convenient gauge is the light-cone gauge, where we impose

$$X^+(\sigma^0,\sigma^1) = x^+ + 2\alpha' p^+ \sigma^0, \tag{16}$$

with the light-cone space-time coordinates

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{D-1}) \tag{17}$$

$$X^- = \frac{1}{\sqrt{2}}(X^0 - X^{D-1}) \tag{18}$$

$$X^I \text{unchanged}, I = 1, ..., D-2. \tag{19}$$

In the light-cone gauge $X^-$, $p^-$ as well as all the operators $\alpha_n^-$ and $\tilde{\alpha}_n^-$ are determined in terms of $p^I$, $\alpha_n^I$ and $\tilde{\alpha}_n^I$, which are the remaining degrees of freedom of the theory.
From now we place ourself in the light cone gauge. The indice $I$ will always refers to a space-time indice going from 1 to $D-2$. What follows is called ligth-cone quantization as opposed to the (equivalent) covariant quantization.

To construct the space of states we first introduce ground states $|k>$, where $k = (k^I)$, defined by the relations

$$p^I|k> = k^I|k>, \quad a_m^I|k> = \tilde{a}_m^I|k> = 0, \quad m > 0, \tag{20}$$

so these states are eigenstates of the total momentum operators and carry no oscillator excitation.
Then a basis $B$ of the space of possible states is obtained by acting with creation operators :

$$B = \{a_{-m_1}^{I_1}...\tilde{a}_{-n_1}^{J_1}|k> \ | \ m_l, n_l > 0\}. \tag{21}$$

As explained in ref. [4, 5, 6, 7] , the Euler-Lagrange equations coming from the Nambu-Goto action lead to the mass-shell condition :

$$M^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(N + a + \tilde{N} + \tilde{a}) \tag{22}$$

where $N$ and $\tilde{N}$ are the number operators definied by

$$N = \sum_{n=1}^{+\infty} n\, a^I_{-n} a_{n\,I} \tag{23}$$

$$\tilde{N} = \sum_{n=1}^{+\infty} n\, \tilde{a}^I_{-n} \tilde{a}_{n\,I}, \tag{24}$$

and $a$ and $\tilde{a}$ are ordering constants comming from the ambiguity of the ordering of creation and annihilation operators. These constants have to satisfy the relation

$$a = \tilde{a} = -\frac{D-2}{24} \tag{25}$$

In order to provide an invariant action under Lorentz transformations ([4]) one can show that we must have $a = \tilde{a} = -1$, so that the dimension of space-time must be $D = 26$. In the covariant quantization procedure the value of the ordering constants and thus the value of the dimension $D = 26$ enables to remove unphysical states from the theory ([5, 6]).

In addition the Euler-Lagrange equation gives the relation

$$N = \tilde{N} \tag{26}$$

which states that the "‘number"’ of left-moving oscillators must be equal to the "‘number"’ of right-moving oscillators.

As we are interested in low energy physics we have to look at the states with lowest energy.
This theory contains states of negative mass-squared, called tachyons, which are a priori not physical and shall be removed by additional constraints.
Then the first excited states are the massless states

$$\sum_{1 \leq I,J \leq D-2} R_{I,J}\, a^I_{-1} \tilde{a}^J_{-1} |k>, \tag{27}$$

where $R_{I,J}$ are the elements of an arbitrary square matrix of size $D-2$.
These states can be split into three groups (ref. [4]):

- $$\sum_{1 \leq I,J \leq D-2} S_{I,J}\, a^I_{-1} \tilde{a}^J_{-1} |k>, \tag{28}$$

  where the matrix $S_{I,J}$ is symetric and traceless. These states correspond exactly to the one particle graviton states one can find in a quantum theory of the free gravitational field.

- $$\sum_{1\leq I,J\leq D-2} A_{I,J}\, a^I_{-1}\tilde{a}^J_{-1}|k>, \tag{29}$$

  where the matrix $A_{I,J}$ is antisymetric. These states correspond to the one particle states of a Kalb-Ramond field, which is an antisymetric tensor field with two indices $B_{\mu\nu}$ (kind of generalization of the Maxwell gauge field $A_\mu$).

- $$S\, a^I_{-1}\tilde{a}^I_{-1}|k>, \quad (I \text{summed from 1 to } D-2), \tag{30}$$

  where $S$ is a constant coming from the decomposition of $R_{I,J}$ into a symetric traceless part, an antisymetric part and a part proportional to the identity matrix. This single state corresponds to a one particle state of a massless scalar field called dillaton field.

As we can see, this construction naturally gives rise to gauge bosons, so that it is called bosonic string theory. In order to obtain fermions, we have to introduce superstrings.

## 2.3 Superstrings

The principle of superstrings is to consider two sets of new space-time variables or fields $\psi^\mu_+(\sigma), \psi^\mu_-(\sigma)$ with the property that they anticommute rather than commute. These two sets of variables compose a space-time spinor with two components $\Psi^\mu$, called Majorana spinor.
The action of the string becomes the RNS action :

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,(\partial_\alpha X^\mu \partial^\alpha X_\mu + i\,\bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu), \tag{31}$$

where $\rho^\alpha$ are the two-dimentional spin matrices

$$\rho^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{32}$$

and $\bar{\Psi}^\mu = \Psi^{\mu\dagger}\rho^0$.
This action is invariant under global world-sheet supersymmetry transformations :

$$\delta X^\mu = \bar{\varepsilon}\Psi^\mu, \quad \delta\Psi^\mu = -i\rho^\alpha \varepsilon \partial_\alpha X^\mu. \tag{33}$$

The equations of motion are the wave equation and the Dirac equation

$$\partial^2 X^\mu = 0, \quad \rho^\alpha \partial_\alpha \Psi^\mu = 0. \tag{34}$$

They imply a decoupling between what appear to be left and right-moving world-sheet fields

$$\psi^\mu_+ = \psi^\mu_+(\sigma^+), \quad \psi^\mu_- = \psi^\mu_-(\sigma^-). \tag{35}$$

$$S = \frac{1}{2\pi\alpha'}\int_{\Sigma'} d\sigma^+ d\sigma^- \,(\partial_+ X^\mu \partial_- X_\mu - i\,\psi_+^\mu \partial_+ \psi_{+\mu} - i\,\psi_-^\mu \partial_- \psi_{-\mu}). \tag{36}$$

Requiring the vanishing of the boundary terms when varying the action leads to two types of boundary conditions for each of the fermion field, then for instance for $\psi_+^\mu$ we have the possibilities :

- the Ramond (or R) boundary conditions which are periodic

$$\psi_+^\mu(\sigma^0, \sigma^1 = \pi) = \psi_+^\mu(\sigma^0, \sigma^1 = 0), \tag{37}$$

- the Neveu-Schwarz (or NS) boundary conditions which are antiperiodic

$$\psi_+^\mu(\sigma^0, \sigma^1 = \pi) = -\psi_+^\mu(\sigma^0, \sigma^1 = 0). \tag{38}$$

Then the equations of motion lead to the following mode expansions :

$$\psi_+^\mu = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in\sigma^+}, \quad \text{for R boundary conditions,} \tag{39}$$

$$\psi_+^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2ir\sigma^+}, \quad \text{for NS boundary conditions,} \tag{40}$$

and similar expressions for right-moving fermion fields, providing another set of mode-oscillators $\tilde{d}_n^\mu, \tilde{b}_r^\mu$.

When we go to quantization of these world-sheet fermion fields, we obtain the anticommutation relations :

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}, \quad \text{in the R-sector,} \tag{41}$$

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}, \quad \text{in the NS-sector,} \tag{42}$$

from the required relations

$$\frac{i}{4\pi\alpha'}\{\psi_+^\mu(\sigma^0, \sigma^1), \psi_+^\nu(\sigma^0, \sigma^{1'})\} = i\delta(\sigma^1 - \sigma^{1'})\eta^{\mu\nu}. \tag{43}$$

(We have similar relations for right-moving fermions)

We obtain a basis of the space of physical states, in the light-cone gauge, by acting on the vacuum with creation operators $a_{-n}^I$, $d_{-m}^I$ or $b_{-r}^I$, $\tilde{d}_n^I$ or $\tilde{b}_r^I$, $n, m, r > 0$, $I = 1, ..., D-2$.

The mass-shell condition becomes

$$M^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(N + a + \tilde{N} + \tilde{a}) \tag{44}$$

where $N$ and $\tilde{N}$ are the number operators definied by

$$N = \sum_{n=1}^{+\infty} n\, a^I_{-n} a_{n\,I} + \sum_{r\in\mathbb{N}+\frac{1}{2}} r\, b^I_{-r} b_{r\,I} \tag{45}$$

$$\tilde{N} = \sum_{n=1}^{+\infty} n\, \tilde{a}^I_{-n} \tilde{a}_{n\,I} + \sum_{r\in\mathbb{N}+\frac{1}{2}} r\, \tilde{b}^I_{-r} \tilde{b}_{r\,I}, \tag{46}$$

for NS boundary conditions in both left and right-moving sectors. For different boundary conditions we have similar expressions involving $d^I_n$ and $\tilde{d}^I_n$.
The ordering constants $a$ and $\tilde{a}$ are given by

$$\begin{aligned} a &= -\frac{D-2}{24} + a_{\psi_+} \\ \tilde{a} &= -\frac{D-2}{24} + a_{\psi_-} \\ a_{\psi_\pm} &= \frac{D-2}{24} \quad \text{R-boundary conditions} \\ &= -\frac{D-2}{48} \quad \text{NS-boundary conditions} \end{aligned} \tag{47}$$

The condition on the dimension D preserving the invariance of the action under Lorentz transformations becomes $D = 10$. In this case we have

$$\begin{aligned} a &= -\frac{1}{3} + a_{\psi_+} \\ \tilde{a} &= -\frac{1}{3} + a_{\psi_-} \\ a_{\psi_\pm} &= \frac{1}{3} \quad (\text{R}) \\ &= -\frac{1}{6} \quad (\text{NS}). \end{aligned} \tag{48}$$

The relation between left and right-moving number operators is now

$$N + a = \tilde{N} + \tilde{a}. \tag{49}$$

One crucial point to notice at this stage is the fact that with R-boundary conditions the oscillators $d^I_0$ and $\tilde{d}^I_0$ act on the vaccum (ground states) without adding any mass, so they degenerate the vaccum. In the rest of this discussion we will forget the total momentum of the string and consider the vacuum to be one ground state $|0>$, which is degenerated under the action of zero mode oscillators in the R-sector in $|s>=|\pm,\pm,...,\pm>$ giving $2^{D-2/2}$ ground states per side (left and/or right) with R-boundary conditions. An explanation of this process can be found in ref. [7].

At this point one can find the low energy states (tachyon, massless, ...) in each of the four possible sectors R-R, R-NS, NS-R, NS-NS, finding space-time bosons (incuding the graviton, dillaton and antisymetric tensor) and space-time fermions, as described in ref. [6].

In order to be closer to the notations of the litterature about heterotic strings, I will adopt from now a slightly different formalism (see ref. [8, 5]) where we consider the couple of variables

$$z = \sigma^0 + i\sigma^1, \quad \bar{z} = \sigma^0 - i\sigma^1, \tag{50}$$

and, by gauge fixing, we obtain the action

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma dz d\bar{z} \, (\partial_z X^\mu \partial_{\bar{z}} X_\mu + i \, \psi^\mu \partial_z \psi_\mu + i\bar{\psi}^\mu \partial_{\bar{z}} \bar{\psi}_\mu), \tag{51}$$

where $\psi^\mu = \psi^\mu(z)$ and $\bar{\psi}_\mu = \bar{\psi}_\mu(\bar{z})$ correspond to left and right-moving fermion fields.

# 3 Free fermionic models

## 3.1 Heterotic string and free fermionic models

In order to get an overview of the construction of the heterotic string in four dimensions, one can refer to [8, 7].

The heterotic string corresponds to a particular construction of the string, where we consider the superstring formalism for the left-moving world-sheet fields and the bosonic formalism for the right-moving world-sheet fields. This means that we only keep the superconformal invariance for the left movers, so that we consider the fields :

$$X_+^\mu(z), \quad \psi^\mu(z), \quad \mu = 0, ..., 9. \tag{52}$$

$$X_-^\mu(\bar{z}), \quad X_-^J(\bar{z}), \quad \mu = 0, ..., 9, \; J = 10, ..., 25. \tag{53}$$

where $z = \tau + i\sigma$, $\bar{z} = \tau - i\sigma$.
This theory implies that the dimension of space-time is $D = 10$. The 16 additional bosonic right-moving fields $X_-^J(\bar{z})$ no longer carry a space-time indice, so we call them internal world-sheet fields.

Now for the specific need of our models, we are going to interpret these 16 bosonic internal fields as fermion fields. This reinterpretation is made possible by the equivalence

$$: e^{iX_-^J}(\bar{z}) := \bar{\lambda}^{2J-1}(\bar{z}) + i\bar{\lambda}^{2J}(\bar{z}), \tag{54}$$

where $\bar{\lambda}^x$ is our notation for a right-moving fermion field, called as well right-moving Majorana-Weyl fermion.
This equivalence is refered as bosonisation or fermionisation in the litterature and is detailed, for instance, in ref. [5].

This construction lets us with a heterotic string in a ten dimensional space-time.
Now we are interested in such a model in a four dimensional spacetime, so what we do is just following the same procedure and restrict the spacetime indice $\mu$ to run only from 0 to 3 and add additionnal free Majorana-Weyl fermionic fields both in the left and right-moving sectors so that the conformal anomaly cancels (we replace missing spacetime coordinates by internal fermionic fields to make the theory coherent). The set of fields we consider in this theory are :

$$\begin{aligned} X^\mu(z, \bar{z}), &\quad \mu = 0, ..., 3 \\ \psi^\mu(z), &\quad \mu = 0, ..., 3 \\ \lambda^i(z), &\quad i = 1, ..., 18 \\ \bar{\lambda}^j(\bar{z}), &\quad j = 1, ..., 44. \end{aligned} \tag{55}$$

The string action takes the form :

$$S = \int d^2z[\partial_z X^\mu \partial_{\bar{z}} X_\mu - 2i\psi^\mu(z)\partial_z\psi_\mu(z) - 2i\lambda^i(z)\partial_z\lambda^i(z) - 2i\bar{\lambda}^j(\bar{z})\partial_{\bar{z}}\bar{\lambda}^j(\bar{z})] \tag{56}$$

The models of particle physics derived from this construction are called free fermionic models.
From now we restrict ourselves to the light-cone gauge, where the field $X^{0,3}$ and $\psi^{0,3}$ can be expressed as a combinaison of the other fields and are no longer degrees of freedom.

## 3.2 Boundary conditions

In order to provide predictions in a perturbative theory, we are interested in the partition function of a one-loop diagram, which corresponds to the world-sheet being a torus (vacuum to vacuum string amplitude). On this world-sheet we have to specify two boundary conditions for the two non-contractible loops of the torus for each (free) fermionic field. These conditions express the shifts of phase of the fermionic fields under parallel transport around a non-contractible loop.

$$f \to -e^{i\pi\alpha(f)} f \tag{57}$$

where f is a fermionic field and the minus sign is conventional.
A set of specified phases for all world-sheet fermions for one non-contractible loop is called a spin structure and it is expressed as a 64 dimensional boundary condition vector. In our notation a generic vector is given by

$$\vec{b} = \{\alpha(\psi^1), \alpha(\psi^2), \alpha(\lambda^1), ..., \alpha(\lambda^{18})|\alpha(\bar{\lambda}^1), ..., \alpha(\bar{\lambda}^{44})\}. \tag{58}$$

Two of these vectors or spin structures specify the boundary conditions for the fermionic fields on the toroidal world-sheet. To two spin structures $\vec{b}_i, \vec{b}_j$ we associate the partition function $Z\binom{\vec{b}_i}{\vec{b}_j}$.

In this report the theory we have developped only consider periodic (R) or antiperiodic (NS) boundary conditions, so $\alpha(f)$ can only take the values 1 (R) or 0 (NS).

Two usefull spin structures are given by the vectors

$$\vec{1} = \{1, 1, ..., 1\} \quad \text{all boundary conditions periodic} \tag{59}$$

$$\vec{0} = \{0, 0, ..., 0\} \quad \text{all boundary conditions antiperiodic} \tag{60}$$

For the bosonic fields $X^\mu$ the theory imposes the boundary conditions to be periodic.

### 3.3 Modular invariance

We would like now to give a general expression for the partition function of a one-loop string amplitude. To do that we need to consider all possible torus. Basically a torus can be seen as a parallelogram whose opposite sides correspond each other, so that for instance the points of the right side are the same as those of the left side. If we parameterize the torus in the complexe plane we need two complexe parameters $\tau_1, \tau_2$ for the two sides of the parallelogram. $z = x\tau_1 + y\tau_2$, $x, y \in [0,1]$, describes the torus$(\tau_1, \tau_2)$.
Actually one can see that, making use of the reparameterization $z \rightarrow \lambda z$, one complexe parameter $\tau$ is only needed since the other can be fixed to 1,

$$z = x + y\tau, \quad x, y \in [0,1]. \tag{61}$$

In addition one can notice that the torus is left invariant by the two following transformations

$$T: \tau \longrightarrow \tau + 1 \quad \text{redefines the torus} \tag{62}$$

$$S: \tau \longrightarrow -1/\tau \quad \text{swaps the two coordinates, reorients the torus} \tag{63}$$

These transformations generate a group of tranformations, called modular group,

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z},\ ab - cd = 1. \tag{64}$$

A function invariant under these transformations is called modular invariant.

The total one-loop partition function then requires to integrate over all independent values of $\tau$,

$$Z = \int_D \frac{d\tau d\bar{\tau}}{Im(\tau)^2} Z(\tau), \tag{65}$$

where D is one region of the complexe plane matching the independent values of $\tau$ and $\frac{d\tau d\bar{\tau}}{Im(\tau)^2}$ is chosen because it is modular invariant.
Then we have to require that $Z(\tau)$ is modular invariant as well, which means that the partition function of a torus does not change because of reparameterization.
Details can be found in ref [7].

### 3.4 ABK rules

The ABK rules are general rules for model building in free fermionic 4D-heterotic theory. They have been developped originally by Antoniadis, Bachas and Kounnas in the papers [1, 3].
These rules come from the modular invariance constraint imposed on the partition function of one-loop diagrams (torus $Z(\tau)$) and multiloop diagrams. If we remember the fermionic construction we have done above, we know that each

possible one-loop amplitude corresponds to two sets of boundary conditions on the torus $\tau$ or spin structures $\vec{\alpha}$, $\vec{\beta}$ (64 components vectors). We can call the associated partition function $Z\binom{\vec{\alpha}}{\vec{\beta}}(\tau)$, so that we have

$$Z(\tau) = \sum_{\text{spin structures } \vec{\alpha},\vec{\beta}} C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} Z\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix}(\tau), \tag{66}$$

where $C\binom{\vec{\alpha}}{\vec{\beta}}$ are complexe phases.

As the 64 fermion fields on the world-sheet (torus) are free, which means without interactions between themselves, the partition function $Z\binom{\vec{\alpha}}{\vec{\beta}}$ is simply the product of the partition functions associated to each fermion field, which can take only four values : $Z\binom{0}{0}(\tau), Z\binom{0}{1}(\tau), Z\binom{1}{0}(\tau), Z\binom{1}{1}(\tau)$. These four functions of $\tau$ have been calculated (see for instance [8, 5])and these calculations show that the modular transformations T and S result in permutations of these four functions (the expressions of these four functions are exchanged by T and S). Thus imposing modular invariance for $Z(\tau)$ results in constraints on the phases $C\binom{\vec{\alpha}}{\vec{\beta}}$, which in some cases must be equal.
The resulting relations, given by ABK, are :

$$C\begin{pmatrix}\vec{\alpha}\\ \vec{\alpha}\end{pmatrix} = -e^{\frac{i\pi\vec{\alpha}.\vec{\alpha}}{4}} C\begin{pmatrix}\vec{\alpha}\\ \vec{1}\end{pmatrix} \tag{67}$$

$$C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} = e^{\frac{i\pi\vec{\alpha}.\vec{\beta}}{2}} C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} \tag{68}$$

$$C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}+\vec{\delta}\end{pmatrix} = e^{i\pi\vec{\alpha}(\psi^\mu)} C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} C\begin{pmatrix}\vec{\alpha}\\ \vec{\delta}\end{pmatrix} \tag{69}$$

$$C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} = \pm 1 \tag{70}$$

where $\alpha(\psi^\mu) = \alpha(\psi^1) = \alpha(\psi^2) = 0$ or 1, and the product $\vec{\alpha}.\vec{\beta}$ is defined by

$$\vec{\alpha}.\vec{\beta} = \left(\frac{1}{2}\sum_{\text{left fermions}} - \frac{1}{2}\sum_{\text{right fermions}}\right) \vec{\alpha}(f)\vec{\beta}(f). \tag{71}$$

When two fermion fields $\lambda^i, \lambda^j$ have the same set of boundary conditions they can be combine in a complex fermion

$$\phi^{ij} = \frac{1}{\sqrt{2}}(\lambda^i + i\lambda^j). \tag{72}$$

We say that one complexe fermion is equivalent to two real fermions. This formalism is usefull for practical calculations.
Then we have

$$\vec{\alpha}.\vec{\beta} = \left(\frac{1}{2}\sum_{\text{left real}} + \sum_{\text{left complex}} - \frac{1}{2}\sum_{\text{right real}} - \sum_{\text{right complex}}\right) \vec{\alpha}(f)\vec{\beta}(f) \tag{73}$$

These relations do not explain how to choose the spin structures, actually they imply that different states corresponding to different spin structures must exist together. To make the theory coherent one has to choose a basis of spin structures $(\vec{b_i})$, $i = 1, ..., N$, then the spin structures corresponding to physical states are all the combinations of these basis vectors, so that the spin structures form a finite group $\Xi$, with the operation

$$\vec{\alpha} + \vec{\beta} = \{\vec{\alpha}(f_1) + \vec{\beta}(f_1), ..., \vec{\alpha}(f_{64}) + \vec{\beta}(f_{64})\} \tag{74}$$

where

$$0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0. \tag{75}$$

Then the choice of the basis $(b_i)$ of $\Xi$ (we now drop the arrow under basis vectors) must satisfy the following conditions :

$$\begin{array}{rcl} \sum_i m_i b_i = 0 & \text{iff} & \forall i,\, m_i = 0 \, mod \, 2 \quad \text{(independence)} \\ \forall i, j, \quad b_i . b_j & = & 0 \, mod \, 2 \\ \forall i, \quad b_i . b_i & = & 0 \, mod \, 4 \\ b_1 = \vec{1} \quad (\vec{1} \in \Xi) & & \end{array} \tag{76}$$

These are not the general ABK rules, but these rules applied to the simplest case where the spin structures contains only 0's and 1's.

To complete this construction we have to impose another set of constraints on physical states called GSO projection, this constraints come from calculations of the partition function, showing that the states contributing to it satisfy these conditions, as detailed in [3]. The GSO projection selects the states $|s>^\alpha$ belonging to the $\alpha$ sector (meaning with $\alpha$ boundary conditions) satisfying

$$\forall i, \quad e^{i\pi b_i F_\alpha} |s>^\alpha = \delta_\alpha C \begin{pmatrix} \alpha \\ b_i \end{pmatrix}^* |s>^\alpha \tag{77}$$

where $\delta_\alpha = e^{i\pi\vec{\alpha}(\psi^\mu)}$,

$$b_i F_\alpha = \left( \sum_{\text{real + complex left}} - \sum_{\text{real + complex right}} \right) b_i(f) F_\alpha(f),$$

and $F_\alpha(f)$ is the fermion number operator

$$\begin{array}{rcl} F_\alpha(f) & = & 1 \\ F_\alpha(f^*) & = & -1 \quad \text{if } f \text{ complex} \\ F_\alpha(|+>) & = & 0 \quad \text{where } |+>=|0> \text{ is the state of a degenerated vacuum} \\ & & \text{without oscillator} \\ F_\alpha(|->) & = & -1 \quad \text{where } |->=f_0^\dagger|0> \text{ is the state of a degenerated vacuum} \\ & & \text{with zero mode oscillator} \end{array} \tag{78}$$

To build a model we have to choose a set of basis vectors (spin structures) satisfying ABK conditions and we have to fix as well a set of free phases $C\binom{b_i}{b_j}$, so we have a lot of free parameters, which is the main problem of free fermionic models.

Before applying these rules to a concrete example, it is crucial to know the mass-shell relation for the string carrying these 64 real fermions, because what we will look at is only the massless states, which are relevant in low-energy physics. Indeed we expect to find all the particles of the SM or MSSM to be massless in this first approach, because the first excited level above the massless states is of the order of the Planck mass, which is far bigger than particle physics masses. The mechanism which gives mass to particles comes from a disconnected theory involving symmetry breaking and studied in details in litterature, but which is not the object of this discussion.
The mass-shell relations for the string come from the relations (44),(49) applied to the case of multiple fermion fields with various boundary conditions around the string. We take D = 4 and for each internal fermion we add $a_\lambda = 1/24$ if it has periodic boundary condition or $a_\lambda = -1/48$ if it has antiperiodic boundary condition. Then we obtain the conditions, taking $\alpha' = 2$ for convienience, in a string sector (or spin structure) $\alpha$ :

$$M_L^2 = -\frac{1}{2} + \frac{\alpha_L.\alpha_L}{8} + \sum_{\text{left fermions}} N(f) \tag{79}$$

$$M_R^2 = -1 + \frac{|\alpha_R.\alpha_R|}{8} + \sum_{\text{right fermions}} \tilde{N}(f) \tag{80}$$

$$M_L^2 = M_R^2 \tag{81}$$

where $\alpha_L$ and $\alpha_R$ are the parts of the vector $\alpha$ corresponding only to the left and right fermions respectively.

This formalism is pretty complicated, so we are going to see a simple example to understand how it works.

The simplest set of basis vectors we can choose is to have only the vector $\vec{1}$ in the basis, which is compatible with ABK conditions (76).
Then we have $\Xi = \{\vec{1}, \vec{0}\}$, so we have physical states in only two sectors.

The states in the $\vec{1}$ sector have mass $M_L^2 > 3/4$ and $M_R^2 > 7/4$, so there is no masseless states in this sector.

In the $\vec{0}$ sector, also called NS-sector, the ground states verify $M_L^2 = M_R^2 = -\frac{1}{2}$ which is negative, so they are tachyonic states and by the way unphysical.These tachyonic states $\bar{\lambda}^j|0>_{NS}$ are obtained by acting on the vacuum $|0>_{NS}$ with one right fermionic creation operator raising $M_R^2$ by one half. This is possible because $\bar{\lambda}^j$ has NS boundary condition (if it had R boundary condition it would have raised $M_R^2$ by 1 and it would not satisfy $M_L^2 = M_R^2$ ). We use here the

same notation for the fermionic field and the creation operator.
Then we have massless states obtained by acting on the vaccum in several ways :

- $\psi^I \partial \bar{X}^J |0>_{NS}$, where $\partial \bar{X}^J$ replaces bosonic creation operator raising $M_R^2$ by 1 and $\psi^I$ replaces fermionic creation operator raising $M_L^2$ by 1/2. These states correspond to the graviton, the dillaton and the antisymmetric tensor we discussed before, but in four dimensions this time ;
- $\psi^I \bar{\lambda}^i \bar{\lambda}^j |0>_{NS}$, $i, j = 1, ..., 44$ (one left-moving and two right-moving creation operators). The fields carrying these states have the same commutation relations as the generators of SO(44), so the states can be identified with the gauge bosons of the gauge group SO(44) (adjoint representation of SO(44));
- $\lambda^k \partial \bar{X}^J |0>_{NS}$, $k = 1, ..., 18$. For similar reasons, these states can be identified with gauge bosons of $SU(2)^6$;
- $\lambda^k \bar{\lambda}^i \bar{\lambda}^j |0>_{NS}$. These states are bosons of scalar fields, charged under $SU(2)^6 \times SO(44)$.

Then we have to perform the GSO projection in this sector (NS-sector) :

- $$\begin{aligned} e^{i\pi \vec{1} F_{\vec{0}}} \psi^I \partial \bar{X}^J |0>_{NS} &= e^{i\pi(1+0)} \psi^I \partial \bar{X}^J |0>_{NS} \\ &= -\psi^I \partial \bar{X}^J |0>_{NS} \\ &= 1 \times (-1) \times \psi^I \partial \bar{X}^J |0>_{NS} \\ &= \delta_{\vec{0}} C \begin{pmatrix} \vec{0} \\ \vec{1} \end{pmatrix}^* \psi^I \partial \bar{X}^J |0>_{NS} \end{aligned}$$

  So the states $\psi^I \partial \bar{X}^J |0>_{NS}$ verify the GSO condition (only one in this case because we have only one basis vector). In this calculation we used $C\binom{\vec{0}}{\vec{1}} = -1$, coming from the relations (69).

- $$\begin{aligned} e^{i\pi \vec{1} F_{\vec{0}}} \psi^I \bar{\lambda}^i \bar{\lambda}^j |0>_{NS} &= e^{i\pi(+1-1-1)} \psi^I \bar{\lambda}^i \bar{\lambda}^j |0>_{NS} \\ &= -\psi^I \bar{\lambda}^i \bar{\lambda}^j |0>_{NS} \\ &= \delta_{\vec{0}} C \begin{pmatrix} \vec{0} \\ \vec{1} \end{pmatrix}^* \psi^I \bar{\lambda}^i \bar{\lambda}^j |0>_{NS} \end{aligned}$$

- $$\begin{aligned} e^{i\pi \vec{1} F_{\vec{0}}} \lambda^k \partial \bar{X}^J |0>_{NS} &= e^{i\pi(+1+0)} \lambda^k \partial \bar{X}^J |0>_{NS} \\ &= -\psi^I \lambda^k \partial \bar{X}^J |0>_{NS} \\ &= \delta_{\vec{0}} C \begin{pmatrix} \vec{0} \\ \vec{1} \end{pmatrix}^* \lambda^k \partial \bar{X}^J |0>_{NS} \end{aligned}$$

- $$\begin{aligned} e^{i\pi\vec{1}F_{\vec{0}}}\lambda^k\bar{\lambda}^i\bar{\lambda}^j|0>_{NS} &= e^{i\pi(+1-1-1)}\lambda^k\bar{\lambda}^i\bar{\lambda}^j|0>_{NS} \\ &= -\lambda^k\bar{\lambda}^i\bar{\lambda}^j|0>_{NS} \\ &= \delta_{\vec{0}}C\binom{\vec{0}}{\vec{1}}^*\lambda^k\bar{\lambda}^i\bar{\lambda}^j|0>_{NS} \end{aligned}$$

All the states we have found survive the GSO projection.
So in this model we have the graviton, the dillaton and the antisymetric tensors (one can show that they are present in every possible model) and we have gauge bosons for the large gauge group $SU(2)^6 \times SO(44)$. This model has actually a lot of weaknesses : the gauge group is too large compared to the SM or GUT gauge groups, there is no fermion so we do not find the three families of the SM, there is no supersymmetry, there are tachyonic states (they survive the GSO projection as well).
To improve our model we need to add new basis vectors, see what are the massless states, perform the GSO projections, then identify the gauge group, which always come from the NS-sector, and try to find states corresponding to the chiral fermions of the standard model. The classification, in terms of matter content and gauge group, of the free fermionic models begun in the eighties and goes on today. The most interesting features of these models are the emergence of gauge groups which are possible candidates for grand unification (SO(10), SU(5)) together with chiral families, the possibility to have $N = 1$ supersymmetry and obviously the presence of the graviton in their spectrum.

We are now going to focus on a specified set of basis vectors and provide a general survey of the particule content of the different models, wich is mainly what I have done in the past few months.

# 4 Classification of $SO(6) \times SO(4)$ models

## 4.1 Basis vectors

The choice of the basis vectors is arbitrary in free fermionic models. However people have shown that the free four-dimensional heterotic fermionic theory is equivalent to a geometrical formulation called $Z_2 \times Z_2$ heterotic orbifold theory where the extra six dimensions are compactified on a 6D-torus, resulting in a 4D effective space-time. Then it appears that some transformations leaving invariant the world-sheet of one-loop amplitudes in orbifold models can be translated into specific choices of basis vectors in free fermionic models. For a better understanding, one can refer to [9, 10, 5]. This choice leads to 12 basis vectors $v_i, i = 1, ..., 12$ (see ref [11]).
Before expressing these vectors we have to introduce new notations for the fermionic fields living on the toroidal world-sheet :

$$\begin{aligned} \lambda^{3i-2} &= \chi^i \\ \lambda^{3i-1} &= y^i \quad , \quad i = 1, ..., 6 \\ \lambda^{3i} &= \omega^i \end{aligned}$$

$$\begin{aligned} \bar{\lambda}^{2j-1} &= \bar{y}^j \\ \bar{\lambda}^{2j} &= \bar{\omega}^j \quad , \quad j = 1, ..., 6 \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{2}}(\bar{\lambda}^{11+2j} + i\bar{\lambda}^{12+2j}) &= \bar{\eta}^j, \quad j = 1, 2, 3 \\ \frac{1}{\sqrt{2}}(\bar{\lambda}^{17+2j} + i\bar{\lambda}^{18+2j}) &= \bar{\psi}^j, \quad j = 1, ..., 5 \\ \frac{1}{\sqrt{2}}(\bar{\lambda}^{27+2j} + i\bar{\lambda}^{28+2j}) &= \bar{\phi}^j, \quad j = 1, ..., 8 \end{aligned}$$

then the list of the different world-sheet fermions is

$$\{\psi^{1,2}, \chi^{1,...,6}, y^{1,...,6}, \omega^{1,...,6} | \bar{y}^{1,...,6}, \bar{\omega}^{1,...,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,...,5}, \bar{\phi}^{1,...,8}\}$$

The 12 basis vectors are

$$\begin{aligned} v_1 = 1 &= \{\psi^\mu,\ \chi^{1,...,6}, y^{1,...,6}, \omega^{1,...,6} | \\ &\quad \bar{y}^{1,...,6}, \bar{\omega}^{1,...,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,...,5}, \bar{\phi}^{1,...,8}\}, \\ v_2 = S &= \{\psi^\mu, \chi^{1,...,6}\}, \\ v_{2+i} = e_i &= \{y^i, \omega^i | \bar{y}^i, \bar{\omega}^i\},\ i = 1, \ldots, 6, \\ v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^1, \bar{\psi}^{1,...,5}\}, \\ v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^2, \bar{\psi}^{1,...,5}\}, \\ v_{11} = z_1 &= \{\bar{\phi}^{1,...,4}\}, \\ v_{12} = z_2 &= \{\bar{\phi}^{5,...,8}\}, \end{aligned} \tag{82}$$

where the fermions between brakets are those with R-boundary conditions (periodic) and those not quoted have NS-boundary conditions (antiperiodic).

Without loss of generality we impose the following GSO phases :

$$c\binom{1}{1} = c\binom{S}{1} = c\binom{S}{e_i} = c\binom{S}{b_m} = c\binom{S}{z_n} = -1, \qquad (83)$$
$$i = 1, ..., 6,\ m = 1, 2,\ n = 1, 2.$$

The models we obtain have N=1 supersymmetry.
Once we have performed GSO projections for massless states we end up in the NS-sector with gauge bosons belonging to the adjoint representation of $SO(10) \times U(1)^3 \times SO(8)^2$ and we can identify $SO(10)$ with the gauge group under which observable matter is charged, because $SO(10)$ contains $SU(3) \times SU(2) \times U(1)$ as a subgroup, and the $U(1)^3 \times SO(8)^2$ is supposed to convey exotic interactions for exotic matter which should not be charged under the observable $SO(10)$ and so would interact with standard matter only by gravitation.

Depending on the choices of GSO coefficients, meaning the phases $C\binom{b_i}{b_j}$, one can obtain a large variety of different matter contents, appearing in several **16** and $\bar{\mathbf{16}}$ spinorial representations of $SO(10)$, meaning families of 16 states charged under $SO(10)$ and transforming in the specific way of a spinorial representation of $SO(10)$. For specific choices of GSO phases one can find three families of chiral matter, with their superparnters. Each family matches a standard model family in terms of hypercharge and electromagnetic charge, that is why we call them chiral families.
A review of these results (and more details) can be found in ref [11].

What we are going to do now is going beyond this classification by adding a new basis vector $\alpha$.This vector splits $SO(10)$ in $SO(6) \times SO(4)$, which could be an interesting candidate for GUT or a just another step towards $SU(3) \times SU(2) \times U(1)$,

$$v_{13} = \alpha = \{\bar{\eta}^{1,2,3}, \bar{\psi}^{1,2,3}, \bar{\phi}^{1,2}\}.$$

We can choose one more GSO phase without loss of generality

$$c\binom{S}{\alpha} = -1 \quad .$$

We will call these models $SO(6) \times SO(4)$ models.

For an introduction to group theory and representations in physics one may see ref [12].

## 4.2 The gauge group

We can identify gauge bosons to the states whose associated fields transform as adjoint representations of groups.

In our case, after calculations, we know that gauge bosons may arise from the following eleven sectors :

$$\mathbf{G} = \{\ 0,\quad z_1,\quad z_2,\quad z_1+z_2,\quad x,\quad \alpha,\quad \alpha+z_1,\quad \delta,\quad \delta+z_1,\quad \delta+z_2,\quad \alpha+x\ \} \tag{84}$$

where

$$x = 1 + S + \sum_{i=1}^{6} e_i + z_1 + z_2 = \{\bar{\eta}^{123}, \bar{\psi}^{12345}\} \tag{85}$$

$$\delta = \alpha + x + z_1 + z_2 = 1 + S + \alpha + \sum_{i=1}^{6} e_i = \{\bar{\psi}^{45}, \bar{\phi}^{345678}\} \tag{86}$$

The 0 sector gauge bosons give rise, idependently of GSO phases, to the gauge group :

$$SO(6) \times SO(4) \times U(1)^3 \times SO(4)_1 \times SO(4)_2 \times SO(8), \tag{87}$$

which can also be written :

$$\begin{aligned} &SO(6)_{obs} \times SU(2)_L \times SU(2)_R \times U(1)_1 \times U(1)_2 \times U(1)_3 \\ &\quad \times SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4 \times SO(8)_{hid}. \end{aligned} \tag{88}$$

We want to identify $SO(6)_{obs} \times SU(2)_L \times SU(2)_R$ to our observable gauge group and the other part to the hidden gauge group. To provide a coherent model, we should impose that no physical states are charged under both observable and hidden gauge groups, however for this survey we will not take such constraint into consideration and provide a global view of possible matter content.

The other sectors quoted in (84) can lead to enhancements of the observable and/or hidden gauge group, depending on the choice of the GSO phases. Enhancement is a procedure where gauge bosons from sectors other than NS-sector give rise to a gauge group or combine with gauge bosons from other sectors to provide a larger gauge group. In a first approach we would like to avoid enhancements because they lead to a larger gauge group whereas all our effort have been put in reducing this gauge group. Avoiding enhancement is made possible by a wise choice of GSO coefficients.

We use a slightly different notation here for GSO phases :

$$C\binom{b_i}{b_j} = e^{i\pi(b_i|b_j)}, \quad (b_i|b_j) = 0, 1 \tag{89}$$

with the properties

$$\begin{aligned} (a_i\,|a_j + a_k\,) &= (a_i\,|a_j\,) + (a_i\,|a_k\,)\ \ ,\forall\ a_i : \{\psi^\mu\} \cap a_i = \emptyset && (90) \\ (a_i\,|a_j\,) &= (a_j\,|a_i\,)\ \ ,\forall\ a_i, a_j : a_i \cdot a_j = 0 \bmod 4 && (91) \end{aligned}$$

Here are the conditions for enhancements in all possible sectors.

- The x sector gauge bosons ($x = \{\bar{\eta}^{123}, \bar{\psi}^{12345}\}$) give rise to enhancement in the observable sector when

$$(e_i|x) = (z_k|x) = 0\,, \quad \forall\, i = 1, ..., 6,\; k = 1, 2. \tag{92}$$

  In this case the enhancement is

$$SO(6)_{obs} \times SU(2)_{L/R} \times U(1)' \Longrightarrow SU(6), \tag{93}$$

  where $U(1)'$ is a combination of the $U(1)_i$.

- The $z_1 + z_2$ sector gauge bosons ($z_1 + z_2 = \{\bar{\phi}^{12345678}\}$) give rise to enhancement in the hidden sector when

$$(e_i|z_1 + z_2) = (b_k|z_1 + z_2) = 0\,, \quad \forall\, i = 1, ..., 6,\; k = 1, 2. \tag{94}$$

  In this case the enhancement is

$$SU(2)_{1/2} \times SU(2)_{3/4} \times SO(8)_{hid} \Longrightarrow SO(12). \tag{95}$$

- The $z_1$ sector gauge bosons ($z_1 = \{\bar{\phi}^{1234}\}$) give rise to different enhancements in the following cases :

  1. when

$$(e_i|z_1) = 0\,, \quad \forall\, i = 1, ..., 6. \tag{96}$$

     we have an enhancement if

$$(z_1|z_2) = 0 \tag{97}$$

     or

$$(z_1|z_2) = 1, \quad (z_1|b_1) = (z_1|b_2) = 0. \tag{98}$$

     In the case when $(z_1|z_2) = 0$ we have both the enhancements

$$SO(6)_{obs} \times SU(2)_{1/2} \times SU(2)_{3/4} \quad \Longrightarrow \quad SO(10) \tag{99}$$
$$SO(4)_{obs} \times SU(2)_{2/1} \times SU(2)_{4/3} \quad \Longrightarrow \quad SO(8) \tag{100}$$

     when $(z_1|b_1) = (z_1|b_2) = 1$ (notice that the two groups do not overlap), or we have one enhancement

$$U_i \times SU(2)_{1/2} \times SU(2)_{3/4} \Longrightarrow SO(6) \tag{101}$$

     corresponding to the three other cases for the $b_m$ phases.
     In the case when $(z_1|z_2) = 1$ and $(z_1|b_1) = (z_1|b_2) = 0$, we have the enhancement

$$SU(2)_{1/2} \times SU(2)_{3/4} \times SO(8)_{hid} \Longrightarrow SO(12). \tag{102}$$

2. when
$$\exists!\, j, (e_j|z_1) = 1 \tag{103}$$
we have an enhancement if
$$(z_1|z_2) = 0 \tag{104}$$
and
$$\begin{cases} (z_1|b_1) = 0 & \text{if } j = 1,2 \\ (z_1|b_2) = 0 & \text{if } j = 3,4 \\ (z_1|b_1) = (z_1|b_2) & \text{if } j = 5,6 \end{cases} \tag{105}$$
In any of these cases the enhancement is
$$SU(2)_{1/2} \times SU(2)_{3/4} \Longrightarrow SO(5). \tag{106}$$

- The $z_2$ sector gauge bosons ($z_2 = \{\bar{\phi}^{5678}\}$) give rise to different enhancements in the following cases :

1. when
$$(e_i|z_2) = 0\,, \quad \forall\, i = 1, ..., 6, \tag{107}$$
we have an enhancement if
$$(z_1|z_2) = 0, \quad (z_2|\alpha) = 1 \tag{108}$$
which corresponds to
$$SO(6)_{obs} \times SO(8)_{hid} \Longrightarrow SO(14) \tag{109}$$
when $(z_2|b_1) = (z_2|b_2) = 1$. And for the other cases for the $b_m$ phases we end up with one of the enhancements
$$U(1)_i \times SO(8)_{hid} \Longrightarrow SO(10). \tag{110}$$
Other conditions for enhancement are
$$(z_1|z_2) = 0, \quad (z_2|\alpha) = 0, \quad (z_2|b_1) = (z_2|b_2) = 1 \tag{111}$$
which leads to
$$SO(4)_{obs} \times SO(8)_{hid} \Longrightarrow SO(12), \tag{112}$$
or
$$(z_1|z_2) = 1, \quad (z_2|b_1) = (z_2|b_2) = 0, \tag{113}$$
which leads to
$$SO(4)_{1/2} \times SO(8)_{hid} \Longrightarrow SO(12), \tag{114}$$

2. when

$$\exists!\, j, (e_j|z_2) = 1, \tag{115}$$

we have an enhancement if

$$(z_1|z_2) = (\alpha|z_2) = 0 \tag{116}$$

and

$$\begin{cases} (z_2|b_1) = 0 & \text{if } j = 1,2 \\ (z_2|b_2) = 0 & \text{if } j = 3,4 \\ (z_2|b_1) = (z_2|b_2) & \text{if } j = 5,6 \end{cases} \tag{117}$$

In any of these cases the enhancement is

$$SO(8)_{hid} \Longrightarrow SO(9). \tag{118}$$

- The $\alpha$ sector gauge bosons ($\alpha = \{\bar{\eta}^{123}, \bar{\psi}^{123}, \bar{\phi}^{12}\}$) give rise to an enhancement when

$$(e_i|\alpha) = (z_2|\alpha) = 0\,, \quad \forall\, i = 1, ..., 6. \tag{119}$$

In this case the enhancement is

$$SO(6)_{obs} \times U(1)' \times SU(2)_{1/2} \Longrightarrow SU(6), \tag{120}$$

where $U(1)'$ is a combination of the $U(1)_i$.

- The $\alpha + z_1$ sector gauge bosons ($\alpha + z_1 = \{\bar{\eta}^{123}, \bar{\psi}^{123}, \bar{\phi}^{34}\}$) give rise to an enhancement when

$$(e_i|\alpha + z_1) = (z_2|\alpha + z_1) = 0\,, \quad \forall\, i = 1, ..., 6. \tag{121}$$

In this case the enhancement is

$$SO(6)_{obs} \times U(1)' \times SU(2)_{3/4} \Longrightarrow SU(6), \tag{122}$$

where $U(1)'$ is a combination of the $U(1)_i$.

- The $\delta$ sector gauge bosons ($\delta = \{\bar{\psi}^{45}, \bar{\phi}^{345678}\}$) give rise to an enhancement when

$$(e_i|\delta) = 0\,, \quad \forall\, i = 1, ..., 6, \tag{123}$$

$$(\delta|b_1) = (\delta|b_2), \tag{124}$$

$$(\delta|\alpha) = 0 \tag{125}$$

and

$$(\delta|b_1) + (\delta|z_1) + (\delta|z_2) + (\delta|1) = 1. \tag{126}$$

In this case the enhancement is

$$SU(2)_{L/R} \times SU(2)_{3/4} \times SO(8)_{hid} \Longrightarrow SO(12). \tag{127}$$

- The $\delta+z_1$ sector gauge bosons ($\delta+z_1=\{\bar{\psi}^{45},\bar{\phi}^{125678}\}$) give rise to an enhancement when

$$(e_i|\delta+z_1)=0\,,\quad \forall\, i=1,...,6, \tag{128}$$

$$(\delta+z_1|b_1)=(\delta+z_1|b_2), \tag{129}$$

$$(\delta+z_1|\alpha)=(\delta+z_1|z_1) \tag{130}$$

and

$$(\delta+z_1|b_1)+(\delta+z_1|z_1)+(\delta+z_1|z_2)+(\delta+z_1|1)=1. \tag{131}$$

In this case the enhancement is

$$SU(2)_{L/R}\times SU(2)_{1/2}\times SO(8)_{hid}\Longrightarrow SO(12). \tag{132}$$

- The $\delta+z_2$ sector gauge bosons ($\delta+z_2=\{\bar{\psi}^{45},\bar{\phi}^{34}\}$) give rise to different enhancements in the following cases :

  1. when
  $$(e_i|\delta+z_2)=0\,,\quad \forall\, i=1,...,6. \tag{133}$$
  we have an enhancement if
  $$(\delta+z_2|\alpha)=1,\quad (\delta+z_2|z_2)=0. \tag{134}$$
  or
  $$\left\{\begin{array}{l}(\delta+z_2|\alpha)=0,\quad (\delta+z_2|b_1)=(\delta+z_2|b_2),\quad (\delta+z_2|z_2)=1\\ (\delta+z_2|z_1)+(\delta+z_2|b_1)=1+(\delta+z_2|1)\end{array}\right. \tag{135}$$
  In the case when $(\delta+z_2|\alpha)=1$ and $(\delta+z_2|z_2)=0$ we have both the enhancements
  $$\begin{aligned} SO(6)_{obs}\times SU(2)_{L/R}\times SU(2)_{3/4} &\Longrightarrow SO(10) \qquad (136)\\ SO(4)_1\times SU(2)_{R/L}\times SU(2)_{4/3} &\Longrightarrow SO(8) \qquad (137)\end{aligned}$$
  when $(\delta+z_2|b_1)=(\delta+z_2|b_2)$ and $(\delta+z_2|b_1+z_1)=(\delta+z_2|1)$ (notice that the two groups do not overlap), or we have one enhancement
  $$U_i\times SU(2)_{L/R}\times SU(2)_{3/4}\Longrightarrow SO(6) \tag{138}$$
  corresponding to the case $(\delta+z_2|b_1)=(\delta+z_2|b_2)$ and $(\delta+z_2|b_1+z_1)=1+(\delta+z_2|1)$ and the cases $(\delta+z_2|b_1)=1+(\delta+z_2|b_2)$.
  In the case when $(\delta+z_2|\alpha)=0$, $(\delta+z_2|z_2)=1$, $(\delta+z_2|b_1)=(\delta+z_2|b_2)$and $(\delta+z_2|b_1+z_1)=1+(\delta+z_2|1)$, we have the enhancement
  $$SU(2)_{L/R}\times SU(2)_{3/4}\times SO(8)_{hid}\Longrightarrow SO(12). \tag{139}$$

2. when
$$\exists!\, j, (e_j|\delta+z_2)=1 \tag{140}$$
we have an enhancement if
$$(\delta+z_2|\alpha)=(\delta+z_2|z_2)=0 \tag{141}$$
and
$$\begin{cases} (\delta+z_2|z_1)+(\delta+z_2|b_1)+(\delta+z_2|1)=0 & \text{if } j=1,2 \\ (\delta+z_2|z_1)+(\delta+z_2|b_2)+(\delta+z_2|1)=0 & \text{if } j=3,4 \\ (\delta+z_2|b_1)=(\delta+z_2|b_2) & \text{if } j=5,6 \end{cases} \tag{142}$$
In any of these cases the enhancement is
$$SU(2)_{L/R}\times SU(2)_{3/4}\Longrightarrow SO(5). \tag{143}$$

- The $\alpha+x$ sector gauge bosons ($\alpha+x=\{\bar\psi^{45},\bar\phi^{12}\}$) give rise to different enhancements in the following cases :

1. when
$$(e_i|\alpha+x)=0\,,\quad \forall\, i=1,...,6. \tag{144}$$
we have an enhancement if
$$(\alpha+x|z_2)=0,\quad (\alpha+x|\alpha)+(\alpha+x|z_1)=1. \tag{145}$$
or
$$\begin{cases} (\alpha+x|z_2)=1, \quad (\alpha+x|\alpha)=(\alpha+x|z_1) \\ (\alpha+x|b_1)=(\alpha+x|b_2), \quad (\alpha+x|b_1)+(\alpha+x|z_1)=1+(\alpha+x|1) \end{cases} \tag{146}$$
In the case when $(\alpha+x|z_2)=0$ and $(\alpha+x|\alpha)+(\alpha+x|z_1)=1$ we have both the enhancements
$$SO(6)_{obs}\times SU(2)_{L/R}\times SU(2)_{1/2} \Longrightarrow SO(10) \tag{147}$$
$$SO(4)_2\times SU(2)_{R/L}\times SU(2)_{2/1} \Longrightarrow SO(8) \tag{148}$$
when $(\alpha+x|b_1)=(\alpha+x|b_2)$ and $(\alpha+x|b_1+z_1)=(\alpha+x|1)$ (notice that the two groups do not overlap), or we have one enhancement
$$U_i\times SU(2)_{L/R}\times SU(2)_{1/2}\Longrightarrow SO(6) \tag{149}$$
corresponding to the case $(\alpha+x|b_1)=(\alpha+x|b_2)$ and $(\alpha+x|b_1+z_1)=1+(\alpha+x|1)$ and the cases $(\alpha+x|b_1)=1+(\alpha+x|b_2)$.
In the case when $(\alpha+x|z_2)=1$, $(\alpha+x|\alpha)=(\alpha+x|z_1)$, $(\alpha+x|b_1)=(\alpha+x|b_2)$ and $(\alpha+x|b_1+z_1)=1+(\alpha+x|1)$, we have the enhancement
$$SU(2)_{L/R}\times SU(2)_{1/2}\times SO(8)_{hid}\Longrightarrow SO(12). \tag{150}$$

2. when

$$\exists!\, j, (e_j|\alpha + x) = 1 \tag{151}$$

we have an enhancement if

$$(\alpha + x|z_2) = 0, \quad (\alpha + x|\alpha) = (\alpha + x|z_1) \tag{152}$$

and

$$\begin{cases} (\alpha + x|z_1) + (\alpha + x|b_1) + (\alpha + x|1) = 0 & \text{if } j = 1,2 \\ (\alpha + x|z_1) + (\alpha + x|b_2) + (\alpha + x|1) = 0 & \text{if } j = 3,4 \\ (\alpha + x|b_1) = (\alpha + x|b_2) & \text{if } j = 5,6 \end{cases} \tag{153}$$

In any of these cases the enhancement is

$$SU(2)_{L/R} \times SU(2)_{1/2} \Longrightarrow SO(5). \tag{154}$$

## 4.3 Twisted matter spectrum and projectors

The massless matter content of these models can arise from two groups of sectors called twisted and untwisted sectors. The twisted sectors are those which may contain chiral matter and so the particles of the MSSM. The untwisted sectors contain other states belonging to vectorial representations of gauge groups.

We will first focus on twisted matter spectrum. After studying the GSO projections we realize that each of these sectors can contain a family 8 states transforming as some representation of a subgroup of the gauge group (88). The conditions for a sector to provide such a family can be expressed in term of projectors $P$(GSO phases) where $P = 0$ means the sector does not provide a family and $P = 1$ means the sector provides a family. These expressions give a convenient summary of GSO projections and enable us to count the number of chiral families very easily for one specific choice of GSO phases. These expressions can be translated into matrix expressions and used to compute the calculations of the number of families, which is actually the real purpose of this work. The matrix expressions of the projectors can be found in the appendix.

For each family we can extract the charges of the different states under the Cartan generators of the gauge group. The Cartan generators are associated with complex world-sheet fermions in this case and we can define for a state $|s>$ in the $b_i$ sector its charge under the complexe fermion $f$ by

$$Q(f) = \frac{1}{2} b_i(f) + F(f, |s>) \tag{155}$$

where $F(f, |s>)$ take values 0, -1 or +1, as defined in (78)and applied to the state $|s>$, but only taking the fermion f into consideration.
The charges of massless states under the different Cartan generators are always $\pm 1/2$ (or 0) in twisted sectors, because these states are actually degenerated

vaccua.
Then we can deduce from these charges for each state its hypercharge and its electromagnetic charge. This provides us a general picture of the chiral matter existing in these models.

Here are the different representations one can find from different sectors with the expressions of the associated projectors.

These representations can take the form $(\mathbf{4},\mathbf{2},\mathbf{1})$, $(\mathbf{4},\mathbf{1},\mathbf{2})$, $(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$, $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$, where $\mathbf{4}$ and $\bar{\mathbf{4}}$ are (anti)spinorial representations of $SO(6)_{obs}$ and $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ are respectively {doublet of $SU(2)_L$, singlet of $SU(2)_R$} and {singlet of $SU(2)_L$, doublet of $SU(2)_R$}, using the decomposition $SO(4) = SU(2)_L \times SU(2)_R$.
The sectors that provide these chiral representations are the following :

$$
\begin{aligned}
B^{(1)}_{pqrs} &= S + b_1 + pe_3 + qe_4 + re_5 + se_6 &(156)\\
&= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4,\\
&\quad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^1, \bar{\psi}^{1..5}\}\\
B^{(2)}_{pqrs} &= S + b_2 + pe_1 + qe_2 + re_5 + se_6 &(157)\\
B^{(3)}_{pqrs} &= S + b_3 + pe_1 + qe_2 + re_3 + se_4 &(158)
\end{aligned}
$$

where $p, q, r, s = 0, 1$; $b_3 = b_1 + b_2 + x = 1 + S + b_1 + b_2 + \sum_{i=1}^{6} e_i + \sum_{n=1}^{2} z_n$.
The explicit expressions for the 48 projectors are

$$
\begin{aligned}
P^{(1)}_{pqrs} &= \frac{1}{16}\left(1 - c\binom{e_1}{B^{(1)}_{pqrs}}\right)\cdot\left(1 - c\binom{e_2}{B^{(1)}_{pqrs}}\right) &(159)\\
&\quad\cdot\left(1 - c\binom{z_1}{B^{(1)}_{pqrs}}\right)\cdot\left(1 - c\binom{z_2}{B^{(1)}_{pqrs}}\right)\\
P^{(2)}_{pqrs} &= \frac{1}{16}\left(1 - c\binom{e_3}{B^{(2)}_{pqrs}}\right)\cdot\left(1 - c\binom{e_4}{B^{(2)}_{pqrs}}\right) &(160)\\
&\quad\cdot\left(1 - c\binom{z_1}{B^{(2)}_{pqrs}}\right)\cdot\left(1 - c\binom{z_2}{B^{(2)}_{pqrs}}\right)\\
P^{(3)}_{pqrs} &= \frac{1}{16}\left(1 - c\binom{e_5}{B^{(3)}_{pqrs}}\right)\cdot\left(1 - c\binom{e_6}{B^{(3)}_{pqrs}}\right) &(161)\\
&\quad\cdot\left(1 - c\binom{z_1}{B^{(3)}_{pqrs}}\right)\cdot\left(1 - c\binom{z_2}{B^{(3)}_{pqrs}}\right)
\end{aligned}
$$

The remaining states have charges $+1/2$ or $-1/2$ under the Cartan generators associated with the complexe fermions $\bar{\psi}^{1,\dots,5}$ ($1/2 = 1/2 + 0$, $-1/2 = 1/2 - 1$). These five charges $Q_1, Q_2, Q_3, Q_4, Q_5$ are of interest because they are charges under the observable gauge group, so they should be linked to hypercharge and electromagnetic charge.

The 8 states we may obtain from one praticular sector have the property that the number of charges -1/2 among $Q_1, Q_2, Q_3$ is either even either odd and the number of charges -1/2 among $Q_4, Q_5$ is also either even either odd. The four possibilies we get correspond to the four possible representations we mentioned above and one can easily verify that each of these representations contains eight states. Now we would like to match these states with particles of the standard model.
Suitable definitions for hyperchage and electromagnetic charge here turn to be

$$Y = \frac{1}{3}(Q_1 + Q_2 + Q_3) + \frac{1}{2}(Q_4 + Q_5) \tag{162}$$

$$Q_{em} = Y + \frac{1}{2}(Q_4 - Q_5) \tag{163}$$

These definitions lead to the following results :

| representation | $\bar{\psi}^{1,2,3}$ | $\bar{\psi}^{4,5}$ | $Y$ | $Q_{em}$ |
|---|---|---|---|---|
| | $(+,+,+)$ | $(+,+)$ | 1 | 1 |
| $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ | $(+,+,+)$ | $(-,-)$ | 0 | 0 |
| | $(+,-,-)$ | $(+,+)$ | 1/3 | 1/3 |
| | $(+,-,-)$ | $(-,-)$ | -2/3 | -2/3 |
| | $(-,-,-)$ | $(-,-)$ | -1 | -1 |
| $(\mathbf{4}, \mathbf{1}, \mathbf{2})$ | $(-,-,-)$ | $(+,+)$ | 0 | 0 |
| | $(+,+,-)$ | $(-,-)$ | -1/3 | -1/3 |
| | $(+,+,-)$ | $(+,+)$ | 2/3 | 2/3 |
| $(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1})$ | $(+,+,+)$ | $(+,-)$ | 1/2 | 1,0 |
| | $(+,-,-)$ | $(+,-)$ | -1/6 | 1/3,-2/3 |
| $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ | $(-,-,-)$ | $(+,-)$ | -1/2 | -1,0 |
| | $(+,+,-)$ | $(+,-)$ | 1/6 | -1/3,2/3 |

So we have found states corresponding to particles of the standard model but they come only by eight whereas we need sixteen states to form a family, so there remains a difficulty.

We can find states in other sectors, with degenerated vacuum under $\psi^i$ and $\phi^i$ oscillators so that they mix what we call the observable and hidden sectors. A physical model should project out these states thanks to a suitable choice of GSO phases.

The states corresponding to the representations $(\mathbf{4}, \mathbf{2}, \mathbf{1})$, $(\mathbf{4}, \mathbf{1}, \mathbf{2})$, $(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1})$, $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ where $\mathbf{4}$ and $\bar{\mathbf{4}}$ are (anti)spinorial representations of the observable SO(6) and the others are representations of the hidden $SU(2)_1 \times SU(2)_2 = SO(4)_1$, arise from the following sectors :

$$
\begin{aligned}
B^{(4)}_{pqrs} &= S+b_1+b_2+\alpha+pe_1+qe_2+re_3+se_4 && (164)\\
&= \{\psi^\mu,\chi^{56},(1-p)y^1\bar{y}^1,p\omega^1\bar{\omega}^1,(1-q)y^2\bar{y}^2,q\omega^2\bar{\omega}^2,\\
&\quad (1-r)y^3\bar{y}^3,r\omega^3\bar{\omega}^3,(1-s)y^4\bar{y}^4,s\omega^4\bar{\omega}^4,\bar{\eta}^3,\bar{\psi}^{123},\bar{\phi}^{12}\}\\
B^{(5)}_{pqrs} &= S+b_1+b_3+\alpha+pe_1+qe_2+re_5+se_6 && (165)\\
B^{(6)}_{pqrs} &= S+b_2+b_3+\alpha+pe_3+qe_4+re_5+se_6 && (166)
\end{aligned}
$$

The 48 GSO projectors associated to these sectors are :

$$
\begin{aligned}
P^{(4)}_{pqrs} &= \frac{1}{16}\left(1-c\begin{pmatrix}e_5\\B^{(4)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix}e_6\\B^{(4)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix}z_2\\B^{(4)}_{pqrs}\end{pmatrix}\right) && (167)\\
&\quad\cdot\left(1-c\begin{pmatrix}\alpha+z_1+x\\B^{(4)}_{pqrs}\end{pmatrix}\right)\\
P^{(5)}_{pqrs} &= \frac{1}{16}\left(1-c\begin{pmatrix}e_3\\B^{(5)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix}e_4\\B^{(5)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix}z_2\\B^{(5)}_{pqrs}\end{pmatrix}\right) && (168)\\
&\quad\cdot\left(1-c\begin{pmatrix}\alpha+z_1+x\\B^{(5)}_{pqrs}\end{pmatrix}\right)\\
P^{(6)}_{pqrs} &= \frac{1}{16}\left(1-c\begin{pmatrix}e_1\\B^{(6)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix}e_2\\B^{(6)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix}z_2\\B^{(6)}_{pqrs}\end{pmatrix}\right) && (169)\\
&\quad\cdot\left(1-c\begin{pmatrix}\alpha+z_1+x\\B^{(6)}_{pqrs}\end{pmatrix}\right)
\end{aligned}
$$

Similar sectors are the $B^{(4,5,6)}_{pqrs}+z_1$ which correspond to the representations $(\mathbf{4},\mathbf{2},\mathbf{1})$, $(\mathbf{4},\mathbf{1},\mathbf{2})$, $(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$, $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ of $SO(6)_{obs}\times SO(4)_2$.
The corresponding GSO projectors are the same as the $P^{(4),(5),(6)}_{pqrs}$ replacing $B^{(4,5,6)}_{pqrs}$ by $B^{(4,5,6)}_{pqrs}+z_1$ and $\alpha$ by $\alpha+z_1$.

The charges of these states, as those of the following sectors we are going to study, do no match the charges of the standard model, which indicates that they should probably be removed of the spectrum of a physical model.

Then we have sectors corresponding to the representations $((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))$, $((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))$, $((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))$ and $((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}))$ for the group $SU(2)^4=SO(4)_{obs}\times SO(4)_1$ :

$$
\begin{aligned}
B^{(7)}_{pqrs} &= B^{(4)}_{pqrs}+x \;=\; S+b_3+\alpha+pe_1+qe_2+re_3+se_4 && (170)\\
&= \{\psi^\mu,\chi^{56},(1-p)y^1\bar{y}^1,p\omega^1\bar{\omega}^1,(1-q)y^2\bar{y}^2,q\omega^2\bar{\omega}^2,\\
&\quad (1-r)y^3\bar{y}^3,r\omega^3\bar{\omega}^3,(1-s)y^4\bar{y}^4,s\omega^4\bar{\omega}^4,\bar{\eta}^{12},\bar{\psi}^{45},\bar{\phi}^{12}\}\\
B^{(8)}_{pqrs} &= B^{(5)}_{pqrs}+x \;=\; S+b_2+\alpha+pe_1+qe_2+re_5+se_6 && (171)\\
B^{(9)}_{pqrs} &= B^{(6)}_{pqrs}+x \;=\; S+b_1+\alpha+pe_3+qe_4+re_5+se_6 && (172)
\end{aligned}
$$

The 48 GSO projectors associated to these sectors are :

$$P^{(7)}_{pqrs} = \frac{1}{8}\left(1-c\binom{e_5}{B^{(7)}_{pqrs}}\right)\cdot\left(1-c\binom{e_6}{B^{(7)}_{pqrs}}\right)\cdot\left(1-c\binom{z_2}{B^{(7)}_{pqrs}}\right) \quad (173)$$

$$P^{(8)}_{pqrs} = \frac{1}{8}\left(1-c\binom{e_3}{B^{(8)}_{pqrs}}\right)\cdot\left(1-c\binom{e_4}{B^{(8)}_{pqrs}}\right)\cdot\left(1-c\binom{z_2}{B^{(8)}_{pqrs}}\right) \quad (174)$$

$$P^{(9)}_{pqrs} = \frac{1}{8}\left(1-c\binom{e_1}{B^{(9)}_{pqrs}}\right)\cdot\left(1-c\binom{e_2}{B^{(9)}_{pqrs}}\right)\cdot\left(1-c\binom{z_2}{B^{(9)}_{pqrs}}\right) \quad (175)$$

Here as well we have the similar $B^{(7,8,9)}_{pqrs} + z_1$ sectors corresponding to similar representations but for $SU(2)^4 = SO(4)_{obs} \times SO(4)_2$, with corresponding GGSO projectors identical to the $P^{(7),(8),(9)}_{pqrs}$ replacing $B^{(7,8,9)}_{pqrs}$ by $B^{(7,8,9)}_{pqrs} + z_1$.

The remaining sectors provide states belonging to representations of the hidden sector.
We have a set of 48 sectors giving representations like $((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))$ of $SU(2)^4 = SO(4)_1 \times SO(4)_2$ :

$$\begin{aligned} B^{(10)}_{pqrs} &= B^{(1)}_{pqrs} + x + z_1 \ = \ S + b_2 + b_3 + pe_3 + qe_4 + re_5 + se_6 + z_1 \quad (176)\\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4,\\ &\quad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^{23}, \bar{\phi}^{1..4}\}\\ B^{(11)}_{pqrs} &= B^{(2)}_{pqrs} + x + z_1 \ = \ S + b_1 + b_3 + pe_1 + qe_2 + re_5 + se_6 + z_1 \quad (177)\\ B^{(12)}_{pqrs} &= B^{(3)}_{pqrs} + x + z_1 \ = \ S + b_1 + b_2 + pe_1 + qe_2 + re_3 + se_4 + z_1 \quad (178)\end{aligned}$$

The associated GGSO projectors are :

$$P^{(10)}_{pqrs} = \frac{1}{8}\left(1-c\binom{e_1}{B^{(10)}_{pqrs}}\right)\cdot\left(1-c\binom{e_2}{B^{(10)}_{pqrs}}\right)\cdot\left(1-c\binom{z_2}{B^{(10)}_{pqrs}}\right) \quad (179)$$

$$P^{(11)}_{pqrs} = \frac{1}{8}\left(1-c\binom{e_3}{B^{(11)}_{pqrs}}\right)\cdot\left(1-c\binom{e_4}{B^{(11)}_{pqrs}}\right)\cdot\left(1-c\binom{z_2}{B^{(11)}_{pqrs}}\right) \quad (180)$$

$$P^{(12)}_{pqrs} = \frac{1}{8}\left(1-c\binom{e_5}{B^{(12)}_{pqrs}}\right)\cdot\left(1-c\binom{e_6}{B^{(12)}_{pqrs}}\right)\cdot\left(1-c\binom{z_2}{B^{(12)}_{pqrs}}\right) \quad (181)$$

(182)

Finally we have 48 sectors corresponding to spinorial $\mathbf{8}$ and antispinorial $\bar{\mathbf{8}}$ representations of $SO(8)$ in the hidden sector :

$$B^{(13)}_{pqrs} = B^{(1)}_{pqrs} + x + z_2 = S + b_2 + b_3 + pe_3 + qe_4 + re_5 + se_6 + z_2 \quad (183)$$

$$= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4, (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^{23}, \bar{\phi}^{5..8}\}$$

$$B^{(14)}_{pqrs} = B^{(2)}_{pqrs} + x + z_2 = S + b_1 + b_3 + pe_1 + qe_2 + re_5 + se_6 + z_2 \quad (184)$$

$$B^{(15)}_{pqrs} = B^{(3)}_{pqrs} + x + z_2 = S + b_1 + b_2 + pe_1 + qe_2 + re_3 + se_4 + z_2 \quad (185)$$

And the GSO projectors are :

$$P^{(13)}_{pqrs} = \frac{1}{16}\left(1 - c\binom{e_1}{B^{(13)}_{pqrs}}\right)\cdot\left(1 - c\binom{e_2}{B^{(13)}_{pqrs}}\right)\cdot\left(1 - c\binom{z_1}{B^{(13)}_{pqrs}}\right) \cdot\left(1 - c\binom{\alpha + x}{B^{(13)}_{pqrs}}\right) \quad (186)$$

$$P^{(14)}_{pqrs} = \frac{1}{16}\left(1 - c\binom{e_3}{B^{(14)}_{pqrs}}\right)\cdot\left(1 - c\binom{e_4}{B^{(14)}_{pqrs}}\right)\cdot\left(1 - c\binom{z_1}{B^{(14)}_{pqrs}}\right) \cdot\left(1 - c\binom{\alpha + x}{B^{(14)}_{pqrs}}\right) \quad (187)$$

$$P^{(15)}_{pqrs} = \frac{1}{16}\left(1 - c\binom{e_5}{B^{(15)}_{pqrs}}\right)\cdot\left(1 - c\binom{e_6}{B^{(15)}_{pqrs}}\right)\cdot\left(1 - c\binom{z_1}{B^{(15)}_{pqrs}}\right) \cdot\left(1 - c\binom{\alpha + x}{B^{(15)}_{pqrs}}\right) \quad (188)$$

## 4.4 Vectorial representations

We can now look at the vectorial representations coming from untwisted sectors. These representations all arise from the 144 sectors :

$$
\begin{aligned}
B^{(1)}_{pqrs}+x &= S+b_1+x+pe_3+qe_4+re_5+se_6 && (189)\\
&= \{\psi^\mu,\chi^{12},(1-p)y^3\bar{y}^3,p\omega^3\bar{\omega}^3,(1-q)y^4\bar{y}^4,q\omega^4\bar{\omega}^4,\\
&\quad (1-r)y^5\bar{y}^5,r\omega^5\bar{\omega}^5,(1-s)y^6\bar{y}^6,s\omega^6\bar{\omega}^6,\bar{\eta}^{23}\}\\
B^{(2)}_{pqrs}+x &= S+b_2+x+pe_1+qe_2+re_5+se_6 && (190)\\
B^{(3)}_{pqrs}+x &= S+b_3+x+pe_1+qe_2+re_3+se_4 && (191)
\end{aligned}
$$

We have the following states and the associated projectors :

- $\{\bar{\psi}^{123}\}|R>^{(i)}_{pqrs}$, $i=1,2,3$, where $|R>^{(i)}_{pqrs}$ is the degenerated Ramond vacuum of the $B^{(i)}_{pqrs}$ sector.
  These states transform as a vectorial representation of SO(6).
  The associated GGSO projectors are :

$$
\begin{aligned}
P^{(i)(\bar{\psi}_{123})}_{pqrs} &= \frac{1}{32}\left(1-c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) && (192)\\
&\quad\cdot\left(1-c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{z_2}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{\alpha+x}{B^{(i)}_{pqrs}+x}\right)
\end{aligned}
$$

- $\{\bar{\psi}^{45}\}|R>^{(i)}_{pqrs}$, $i=1,2,3$, where $|R>^{(i)}_{pqrs}$ is the degenerated Ramond vacuum of the $B^{(i)}_{pqrs}$ sector.
  These states transform as a vectorial representation of SO(4).
  The associated GGSO projectors are :

$$
\begin{aligned}
P^{(i)(\bar{\psi}_{45})}_{pqrs} &= \frac{1}{32}\left(1-c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) && (193)\\
&\quad\cdot\left(1-c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{z_2}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1+c\binom{\alpha+x}{B^{(i)}_{pqrs}+x}\right)
\end{aligned}
$$

- $\{\bar{\phi}^{12}\}|R>^{(i)}_{pqrs}$, $i=1,2,3$.
  These states transform as a vectorial representation of SO(4).
  The associated GGSO projectors are :

$$
\begin{aligned}
P^{(i)(\bar{\phi}_{12})}_{pqrs} &= \frac{1}{32}\left(1-c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) && (194)\\
&\quad\cdot\left(1+c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{z_2}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1+c\binom{\alpha+x}{B^{(i)}_{pqrs}+x}\right)
\end{aligned}
$$

- $\{\bar{\phi}^{34}\}|R>^{(i)}_{pqrs}$, $i=1,2,3$.
  These states transform as a vectorial representation of SO(4).
  The associated GGSO projectors are :

$$
\begin{aligned}
P^{(i)(\bar{\phi}_{34})}_{pqrs} &= \frac{1}{32}\left(1-c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) \\
&\cdot\left(1+c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{z_2}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{\alpha+x}{B^{(i)}_{pqrs}+x}\right)
\end{aligned}
\tag{195}
$$

- $\{\bar{\phi}^{5..8}\}|R>^{(i)}_{pqrs}$, $i=1,2,3$.
  These states transform as a vectorial representation of SO(8).
  The associated GGSO projectors are :

$$
\begin{aligned}
P^{(i)(\bar{\phi}_{5..8})}_{pqrs} &= \frac{1}{32}\left(1-c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) \\
&\cdot\left(1-c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1+c\binom{z_2}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1-c\binom{\alpha+x}{B^{(i)}_{pqrs}+x}\right)
\end{aligned}
\tag{196}
$$

- the remaining states in those sectors do not provide vectorial representations.

If we look at the expressions of projectors for twisted matter spectrum and compare with projectors of vectorial representations, we can find a lot of similarities. Actually we can find a matching or isomorphism between the different models which reverse the number of vectorial representations of one type with the number of spinorial (or chiral) representations of some $B^i$ sectors. This kind of duality in the space of possible models has been studied for free fermionic models with only the first 12 vectors $v_i$ in the basis in paper [11], where it is called Spinor-Vector duality. In our case the duality is more complicated and more mysterious. It might be matter for future work. The details of this "'duality"' are given in the appendix.

## 5 Conclusions

We have now developped some usefull tools to translate ABK rules into more direct expressions for $SO(6) \times SO(4)$ models. We know what are their possible gauge groups and their possible matter contents. We found a vast amount of different models providing chiral families. Each family contains only eight states so, in future work, we might find some rule to regroup them in pairs, in order to obtain the families of the MSSM. We might as well try to search for rules to select models (i.e. GSO phases) which provide the right number of families.
A better understanding could be given by the statistical analysis of these models on computer. The expectations we might have as for the results of the computation of these calculations are not straightforward. One interesting feature will be given by looking at the percentage of models giving rise to spectrums whith specific number of chiral families. This could reveal properties common to a majority of models. We can hope that the features we will understand from this analysis will give us a hint as for the contraints we have to impose to select models.

Obviously the analysis of $SO(6) \times SO(4)$ models does not provide a classification of all 4D-heterotic free fermionic models. So future work will be to perform the same classification for other gauge groups, like $SU(5) \times U(1)$ which is a subgroup of $SO(10)$ and a good candidate for grand unification.

# 6 Appendix

## 6.1 Matrix formalism

The projectors can be written as system of equations (one per plane)

$$\Delta_8^{(I)}\, U_8^{(I)} = Y_8^{(I)}\ ,\ I = 1, ..., 15, (+4', 5', ..., 9') \tag{197}$$

$$\Delta_v^{(J)}\, U_{\mathbf{osc}}^{(J)} = Y_{\mathbf{osc}}^{(J)}\ ,\ J = 1, 2, 3,\ \mathbf{osc} = \bar{\psi}^{123}, \bar{\psi}^{45}, \bar{\phi}^{12}, \bar{\phi}^{34}, \bar{\phi}^{5..8}. \tag{198}$$

where the unknowns are the fixed point labels

$$U_8^{(I)} = \begin{bmatrix} p_8^I \\ q_8^I \\ r_8^I \\ s_8^I \end{bmatrix}\ ,\ U_{\mathbf{osc}}^{(J)} = \begin{bmatrix} p_{\mathbf{osc}}^J \\ q_{\mathbf{osc}}^J \\ r_{\mathbf{osc}}^J \\ s_{\mathbf{osc}}^J \end{bmatrix} \tag{199}$$

and

$$\begin{aligned}
\Delta_8^{(1)} &= \begin{bmatrix} (e_1\,|e_3\,) & (e_1\,|e_4\,) & (e_1\,|e_5\,) & (e_1\,|e_6\,) \\ (e_2\,|e_3\,) & (e_2\,|e_4\,) & (e_2\,|e_5\,) & (e_2\,|e_6\,) \\ (z_1\,|e_3\,) & (z_1\,|e_4\,) & (z_1\,|e_5\,) & (z_1\,|e_6\,) \\ (z_2\,|e_3\,) & (z_2\,|e_4\,) & (z_2\,|e_5\,) & (z_2\,|e_6\,) \end{bmatrix} \\
\Delta_8^{(2)} &= \begin{bmatrix} (e_3\,|e_1\,) & (e_3\,|e_2\,) & (e_3\,|e_5\,) & (e_3\,|e_6\,) \\ (e_4\,|e_1\,) & (e_4\,|e_2\,) & (e_4\,|e_5\,) & (e_4\,|e_6\,) \\ (z_1\,|e_1\,) & (z_1\,|e_2\,) & (z_1\,|e_5\,) & (z_1\,|e_6\,) \\ (z_2\,|e_1\,) & (z_2\,|e_2\,) & (z_2\,|e_5\,) & (z_2\,|e_6\,) \end{bmatrix} \\
\Delta_8^{(3)} &= \begin{bmatrix} (e_5\,|e_1\,) & (e_5\,|e_2\,) & (e_5\,|e_3\,) & (e_5\,|e_4\,) \\ (e_6\,|e_1\,) & (e_6\,|e_2\,) & (e_6\,|e_3\,) & (e_6\,|e_4\,) \\ (z_1\,|e_1\,) & (z_1\,|e_2\,) & (z_1\,|e_3\,) & (z_1\,|e_4\,) \\ (z_2\,|e_1\,) & (z_2\,|e_2\,) & (z_2\,|e_3\,) & (z_2\,|e_4\,) \end{bmatrix}
\end{aligned} \tag{200}$$

$$Y_8^{(1)} = \begin{bmatrix} (e_1\,|b_1\,) \\ (e_2\,|b_1\,) \\ (z_1\,|b_1\,) \\ (z_2\,|b_1\,) \end{bmatrix}\ ,\ Y_8^{(2)} = \begin{bmatrix} (e_3\,|b_2\,) \\ (e_4\,|b_2\,) \\ (z_1\,|b_2\,) \\ (z_2\,|b_2\,) \end{bmatrix}\ ,\ Y_8^{(3)} = \begin{bmatrix} (e_5\,|b_3\,) \\ (e_6\,|b_3\,) \\ (z_1\,|b_3\,) \\ (z_2\,|b_3\,) \end{bmatrix} \tag{201}$$

$$\Delta_8^{(4)} = \begin{bmatrix} (e_5\,|e_1) & (e_5\,|e_2) & (e_5\,|e_3) & (e_5\,|e_4) \\ (e_6\,|e_1) & (e_6\,|e_2) & (e_6\,|e_3) & (e_6\,|e_4) \\ (\alpha+z_1+x\,|e_1) & (\alpha+z_1+x\,|e_2) & (\alpha+z_1+x\,|e_3) & (\alpha+z_1+x\,|e_4) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_3) & (z_2\,|e_4) \end{bmatrix}$$

$$\Delta_8^{(5)} = \begin{bmatrix} (e_3\,|e_1) & (e_3\,|e_2) & (e_3\,|e_5) & (e_3\,|e_6) \\ (e_4\,|e_1) & (e_4\,|e_2) & (e_4\,|e_5) & (e_4\,|e_6) \\ (\alpha+z_1+x\,|e_1) & (\alpha+z_1+x\,|e_2) & (\alpha+z_1+x\,|e_5) & (\alpha+z_1+x\,|e_6) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_5) & (z_2\,|e_6) \end{bmatrix} \quad (202)$$

$$\Delta_8^{(6)} = \begin{bmatrix} (e_1\,|e_3) & (e_1\,|e_4) & (e_1\,|e_5) & (e_1\,|e_6) \\ (e_2\,|e_3) & (e_2\,|e_4) & (e_2\,|e_5) & (e_2\,|e_6) \\ (\alpha+z_1+x\,|e_3) & (\alpha+z_1+x\,|e_4) & (\alpha+z_1+x\,|e_5) & (\alpha+z_1+x\,|e_6) \\ (z_2\,|e_3) & (z_2\,|e_4) & (z_2\,|e_5) & (z_2\,|e_6) \end{bmatrix}$$

$$Y_8^{(4)} = \begin{bmatrix} (e_5\,|b_1+b_2+\alpha) \\ (e_6\,|b_1+b_2+\alpha) \\ (\alpha+z_1+x\,|b_1+b_2+\alpha) \\ (z_2\,|b_1+b_2+\alpha) \end{bmatrix} , \; Y_8^{(5)} = \begin{bmatrix} (e_3\,|b_1+b_3+\alpha) \\ (e_4\,|b_1+b_3+\alpha) \\ (\alpha+z_1+x\,|b_1+b_3+\alpha) \\ (z_2\,|b_1+b_3+\alpha) \end{bmatrix} , \quad (203)$$

$$Y_8^{(6)} = \begin{bmatrix} (e_1\,|b_2+b_3+\alpha) \\ (e_2\,|b_2+b_3+\alpha) \\ (\alpha+z_1+x\,|b_2+b_3+\alpha) \\ (z_2\,|b_2+b_3+\alpha) \end{bmatrix}$$

$$\Delta_8^{(4')} = \begin{bmatrix} (e_5\,|e_1) & (e_5\,|e_2) & (e_5\,|e_3) & (e_5\,|e_4) \\ (e_6\,|e_1) & (e_6\,|e_2) & (e_6\,|e_3) & (e_6\,|e_4) \\ (\alpha+x\,|e_1) & (\alpha+x\,|e_2) & (\alpha+x\,|e_3) & (\alpha+x\,|e_4) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_3) & (z_2\,|e_4) \end{bmatrix}$$

$$\Delta_8^{(5')} = \begin{bmatrix} (e_3\,|e_1) & (e_3\,|e_2) & (e_3\,|e_5) & (e_3\,|e_6) \\ (e_4\,|e_1) & (e_4\,|e_2) & (e_4\,|e_5) & (e_4\,|e_6) \\ (\alpha+x\,|e_1) & (\alpha+x\,|e_2) & (\alpha+x\,|e_5) & (\alpha+x\,|e_6) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_5) & (z_2\,|e_6) \end{bmatrix} \quad (204)$$

$$\Delta_8^{(6')} = \begin{bmatrix} (e_1\,|e_3) & (e_1\,|e_4) & (e_1\,|e_5) & (e_1\,|e_6) \\ (e_2\,|e_3) & (e_2\,|e_4) & (e_2\,|e_5) & (e_2\,|e_6) \\ (\alpha+x\,|e_3) & (\alpha+x\,|e_4) & (\alpha+x\,|e_5) & (\alpha+x\,|e_6) \\ (z_2\,|e_3) & (z_2\,|e_4) & (z_2\,|e_5) & (z_2\,|e_6) \end{bmatrix}$$

$$Y_8^{(4')} = \begin{bmatrix} (e_5\,|b_1+b_2+\alpha+z_1\,) \\ (e_6\,|b_1+b_2+\alpha+z_1\,) \\ (\alpha+x\,|b_1+b_2+\alpha+z_1\,) \\ (z_2\,|b_1+b_2+\alpha+z_1\,) \end{bmatrix} \quad , \; Y_8^{(5')} = \begin{bmatrix} (e_3\,|b_1+b_3+\alpha+z_1\,) \\ (e_4\,|b_1+b_3+\alpha+z_1\,) \\ (\alpha+x\,|b_1+b_3+\alpha+z_1\,) \\ (z_2\,|b_1+b_3+\alpha+z_1\,) \end{bmatrix} , \tag{205}$$

$$Y_8^{(6')} = \begin{bmatrix} (e_1\,|b_2+b_3+\alpha+z_1\,) \\ (e_2\,|b_2+b_3+\alpha+z_1\,) \\ (\alpha+x\,|b_2+b_3+\alpha+z_1\,) \\ (z_2\,|b_2+b_3+\alpha+z_1\,) \end{bmatrix}$$

$$\begin{aligned} \Delta_8^{(7)} &= \begin{bmatrix} (e_5\,|e_1\,) & (e_5\,|e_2\,) & (e_5\,|e_3\,) & (e_5\,|e_4\,) \\ (e_6\,|e_1\,) & (e_6\,|e_2\,) & (e_6\,|e_3\,) & (e_6\,|e_4\,) \\ (z_2\,|e_1\,) & (z_2\,|e_2\,) & (z_2\,|e_3\,) & (z_2\,|e_4\,) \end{bmatrix} \\ \Delta_8^{(8)} &= \begin{bmatrix} (e_3\,|e_1\,) & (e_3\,|e_2\,) & (e_3\,|e_5\,) & (e_3\,|e_6\,) \\ (e_4\,|e_1\,) & (e_4\,|e_2\,) & (e_4\,|e_5\,) & (e_4\,|e_6\,) \\ (z_2\,|e_1\,) & (z_2\,|e_2\,) & (z_2\,|e_5\,) & (z_2\,|e_6\,) \end{bmatrix} \\ \Delta_8^{(9)} &= \begin{bmatrix} (e_1\,|e_3\,) & (e_1\,|e_4\,) & (e_1\,|e_5\,) & (e_1\,|e_6\,) \\ (e_2\,|e_3\,) & (e_2\,|e_4\,) & (e_2\,|e_5\,) & (e_2\,|e_6\,) \\ (z_2\,|e_3\,) & (z_2\,|e_4\,) & (z_2\,|e_5\,) & (z_2\,|e_6\,) \end{bmatrix} \end{aligned} \tag{206}$$

$$Y_8^{(7)} = \begin{bmatrix} (e_5\,|b_3+\alpha\,) \\ (e_6\,|b_3+\alpha\,) \\ (z_2\,|b_3+\alpha\,) \end{bmatrix} \quad , \; Y_8^{(8)} = \begin{bmatrix} (e_3\,|b_2+\alpha\,) \\ (e_4\,|b_2+\alpha\,) \\ (z_2\,|b_2+\alpha\,) \end{bmatrix} \; , \; Y_8^{(9)} = \begin{bmatrix} (e_1\,|b_1+\alpha\,) \\ (e_2\,|b_1+\alpha\,) \\ (z_2\,|b_1+\alpha\,) \end{bmatrix} \tag{207}$$

$$\begin{aligned} \Delta_8^{(7')} &= \begin{bmatrix} (e_5\,|e_1\,) & (e_5\,|e_2\,) & (e_5\,|e_3\,) & (e_5\,|e_4\,) \\ (e_6\,|e_1\,) & (e_6\,|e_2\,) & (e_6\,|e_3\,) & (e_6\,|e_4\,) \\ (z_2\,|e_1\,) & (z_2\,|e_2\,) & (z_2\,|e_3\,) & (z_2\,|e_4\,) \end{bmatrix} = \Delta_8^{(7)} \\ \Delta_8^{(8')} &= \begin{bmatrix} (e_3\,|e_1\,) & (e_3\,|e_2\,) & (e_3\,|e_5\,) & (e_3\,|e_6\,) \\ (e_4\,|e_1\,) & (e_4\,|e_2\,) & (e_4\,|e_5\,) & (e_4\,|e_6\,) \\ (z_2\,|e_1\,) & (z_2\,|e_2\,) & (z_2\,|e_5\,) & (z_2\,|e_6\,) \end{bmatrix} = \Delta_8^{(8)} \\ \Delta_8^{(9')} &= \begin{bmatrix} (e_1\,|e_3\,) & (e_1\,|e_4\,) & (e_1\,|e_5\,) & (e_1\,|e_6\,) \\ (e_2\,|e_3\,) & (e_2\,|e_4\,) & (e_2\,|e_5\,) & (e_2\,|e_6\,) \\ (z_2\,|e_3\,) & (z_2\,|e_4\,) & (z_2\,|e_5\,) & (z_2\,|e_6\,) \end{bmatrix} = \Delta_8^{(9)} \end{aligned} \tag{208}$$

$$Y_8^{(7')} = \begin{bmatrix} (e_5\,|b_3+\alpha+z_1\,) \\ (e_6\,|b_3+\alpha+z_1\,) \\ (z_2\,|b_3+\alpha+z_1\,) \end{bmatrix} \quad , \; Y_8^{(8')} = \begin{bmatrix} (e_3\,|b_2+\alpha+z_1\,) \\ (e_4\,|b_2+\alpha+z_1\,) \\ (z_2\,|b_2+\alpha+z_1\,) \end{bmatrix} , \tag{209}$$

$$Y_8^{(9')} = \begin{bmatrix} (e_1\,|b_1+\alpha+z_1\,) \\ (e_2\,|b_1+\alpha+z_1\,) \\ (z_2\,|b_1+\alpha+z_1\,) \end{bmatrix}$$

$$\Delta_8^{(10)} = \begin{bmatrix} (e_1\,|e_3) & (e_1\,|e_4) & (e_1\,|e_5) & (e_1\,|e_6) \\ (e_2\,|e_3) & (e_2\,|e_4) & (e_2\,|e_5) & (e_2\,|e_6) \\ (z_2\,|e_3) & (z_2\,|e_4) & (z_2\,|e_5) & (z_2\,|e_6) \end{bmatrix} = \Delta_8^{(9)}$$

$$\Delta_8^{(11)} = \begin{bmatrix} (e_3\,|e_1) & (e_3\,|e_2) & (e_3\,|e_5) & (e_3\,|e_6) \\ (e_4\,|e_1) & (e_4\,|e_2) & (e_4\,|e_5) & (e_4\,|e_6) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_5) & (z_2\,|e_6) \end{bmatrix} = \Delta_8^{(8)} \qquad (210)$$

$$\Delta_8^{(12)} = \begin{bmatrix} (e_5\,|e_1) & (e_5\,|e_2) & (e_5\,|e_3) & (e_5\,|e_4) \\ (e_6\,|e_1) & (e_6\,|e_2) & (e_6\,|e_3) & (e_6\,|e_4) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_3) & (z_2\,|e_4) \end{bmatrix} = \Delta_8^{(7)}$$

$$Y_8^{(10)} = \begin{bmatrix} (e_1\,|b_2+b_3+z_1) \\ (e_2\,|b_2+b_3+z_1) \\ (z_2\,|b_2+b_3+z_1) \end{bmatrix} ,\ Y_8^{(11)} = \begin{bmatrix} (e_3\,|b_1+b_3+z_1) \\ (e_4\,|b_1+b_3+z_1) \\ (z_2\,|b_1+b_3+z_1) \end{bmatrix} , \qquad (211)$$

$$Y_8^{(12)} = \begin{bmatrix} (e_5\,|b_1+b_2+z_1) \\ (e_6\,|b_1+b_2+z_1) \\ (z_2\,|b_1+b_2+z_1) \end{bmatrix}$$

$$\Delta_8^{(13)} = \begin{bmatrix} (e_1\,|e_3) & (e_1\,|e_4) & (e_1\,|e_5) & (e_1\,|e_6) \\ (e_2\,|e_3) & (e_2\,|e_4) & (e_2\,|e_5) & (e_2\,|e_6) \\ (z_1\,|e_3) & (z_1\,|e_4) & (z_1\,|e_5) & (z_1\,|e_6) \\ (\alpha+x\,|e_3) & (\alpha+x\,|e_4) & (\alpha+x\,|e_5) & (\alpha+x\,|e_6) \end{bmatrix}$$

$$\Delta_8^{(14)} = \begin{bmatrix} (e_3\,|e_1) & (e_3\,|e_2) & (e_3\,|e_5) & (e_3\,|e_6) \\ (e_4\,|e_1) & (e_4\,|e_2) & (e_4\,|e_5) & (e_4\,|e_6) \\ (z_1\,|e_1) & (z_1\,|e_2) & (z_1\,|e_5) & (z_1\,|e_6) \\ (\alpha+x\,|e_1) & (\alpha+x\,|e_2) & (\alpha+x\,|e_5) & (\alpha+x\,|e_6) \end{bmatrix} \qquad (212)$$

$$\Delta_8^{(15)} = \begin{bmatrix} (e_5\,|e_1) & (e_5\,|e_2) & (e_5\,|e_3) & (e_5\,|e_4) \\ (e_6\,|e_1) & (e_6\,|e_2) & (e_6\,|e_3) & (e_6\,|e_4) \\ (z_1\,|e_1) & (z_1\,|e_1) & (z_1\,|e_3) & (z_1\,|e_4) \\ (\alpha+x\,|e_1) & (\alpha+x\,|e_2) & (\alpha+x\,|e_3) & (\alpha+x\,|e_4) \end{bmatrix}$$

$$Y_8^{(13)} = \begin{bmatrix} (e_1\,|b_2+b_3+z_2) \\ (e_2\,|b_2+b_3+z_2) \\ (z_1\,|b_2+b_3+z_2) \\ (\alpha+x\,|b_2+b_3+z_2) \end{bmatrix} ,\ Y_8^{(14)} = \begin{bmatrix} (e_3\,|b_1+b_3+z_2) \\ (e_4\,|b_1+b_3+z_2) \\ (z_1\,|b_1+b_3+z_2) \\ (\alpha+x\,|b_1+b_3+z_2) \end{bmatrix} , \qquad (213)$$

$$Y_8^{(15)} = \begin{bmatrix} (e_5\,|b_1+b_2+z_2) \\ (e_6\,|b_1+b_2+z_2) \\ (z_1\,|b_1+b_2+z_2) \\ (\alpha+x\,|b_1+b_2+z_2) \end{bmatrix}$$

and for vectorials :

$$
\begin{aligned}
\Delta_v^{(1)} &= \begin{bmatrix} (e_1\,|e_3) & (e_1\,|e_4) & (e_1\,|e_5) & (e_1\,|e_6) \\ (e_2\,|e_3) & (e_2\,|e_4) & (e_2\,|e_5) & (e_2\,|e_6) \\ (z_1\,|e_3) & (z_1\,|e_4) & (z_1\,|e_5) & (z_1\,|e_6) \\ (z_2\,|e_3) & (z_2\,|e_4) & (z_2\,|e_5) & (z_2\,|e_6) \\ (\alpha+x\,|e_3) & (\alpha+x\,|e_4) & (\alpha+x\,|e_5) & (\alpha+x\,|e_6) \end{bmatrix} \\
\Delta_v^{(2)} &= \begin{bmatrix} (e_3\,|e_1) & (e_3\,|e_2) & (e_3\,|e_5) & (e_3\,|e_6) \\ (e_4\,|e_1) & (e_4\,|e_2) & (e_4\,|e_5) & (e_4\,|e_6) \\ (z_1\,|e_1) & (z_1\,|e_2) & (z_1\,|e_5) & (z_1\,|e_6) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_5) & (z_2\,|e_6) \\ (\alpha+x\,|e_1) & (\alpha+x\,|e_2) & (\alpha+x\,|e_5) & (\alpha+x\,|e_6) \end{bmatrix} \qquad (214) \\
\Delta_v^{(3)} &= \begin{bmatrix} (e_5\,|e_1) & (e_5\,|e_2) & (e_5\,|e_3) & (e_5\,|e_4) \\ (e_6\,|e_1) & (e_6\,|e_2) & (e_6\,|e_3) & (e_6\,|e_4) \\ (z_1\,|e_1) & (z_1\,|e_2) & (z_1\,|e_3) & (z_1\,|e_4) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_3) & (z_2\,|e_4) \\ (\alpha+x\,|e_1) & (\alpha+x\,|e_2) & (\alpha+x\,|e_3) & (\alpha+x\,|e_4) \end{bmatrix}
\end{aligned}
$$

$$
Y^{(1)}_{\bar\psi^{123}} = \begin{bmatrix} (e_1\,|b_1+x) \\ (e_2\,|b_1+x) \\ (z_1\,|b_1+x) \\ (z_2\,|b_1+x) \\ (\alpha+x\,|b_1+x) \end{bmatrix},\ Y^{(2)}_{\bar\psi^{123}} = \begin{bmatrix} (e_3\,|b_2+x) \\ (e_4\,|b_2+x) \\ (z_1\,|b_2+x) \\ (z_2\,|b_2+x) \\ (\alpha+x\,|b_2+x) \end{bmatrix},\ Y^{(3)}_{\bar\psi^{123}} = \begin{bmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ (z_1\,|b_3+x) \\ (z_2\,|b_3+x) \\ (\alpha+x\,|b_3+x) \end{bmatrix} \qquad (215)
$$

$$
Y^{(1)}_{\bar\psi^{45}} = \begin{bmatrix} (e_1\,|b_1+x) \\ (e_2\,|b_1+x) \\ (z_1\,|b_1+x) \\ (z_2\,|b_1+x) \\ 1+(\alpha+x\,|b_1+x) \end{bmatrix},\ Y^{(2)}_{\bar\psi^{45}} = \begin{bmatrix} (e_3\,|b_2+x) \\ (e_4\,|b_2+x) \\ (z_1\,|b_2+x) \\ (z_2\,|b_2+x) \\ 1+(\alpha+x\,|b_2+x) \end{bmatrix}, \qquad (216)
$$

$$
Y^{(3)}_{\bar\psi^{45}} = \begin{bmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ (z_1\,|b_3+x) \\ (z_2\,|b_3+x) \\ 1+(\alpha+x\,|b_3+x) \end{bmatrix}
$$

$$
Y^{(1)}_{\bar\phi^{12}} = \begin{bmatrix} (e_1\,|b_1+x) \\ (e_2\,|b_1+x) \\ 1+(z_1\,|b_1+x) \\ (z_2\,|b_1+x) \\ 1+(\alpha+x\,|b_1+x) \end{bmatrix},\ Y^{(2)}_{\bar\phi^{12}} = \begin{bmatrix} (e_3\,|b_2+x) \\ (e_4\,|b_2+x) \\ 1+(z_1\,|b_2+x) \\ (z_2\,|b_2+x) \\ 1+(\alpha+x\,|b_2+x) \end{bmatrix}, \qquad (217)
$$

$$
Y^{(3)}_{\bar\phi^{12}} = \begin{bmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ 1+(z_1\,|b_3+x) \\ (z_2\,|b_3+x) \\ 1+(\alpha+x\,|b_3+x) \end{bmatrix}
$$

$$Y^{(1)}_{\bar\phi^{34}} = \begin{bmatrix} (e_1\,|b_1+x) \\ (e_2\,|b_1+x) \\ 1+(z_1\,|b_1+x) \\ (z_2\,|b_1+x) \\ (\alpha+x\,|b_1+x) \end{bmatrix} \;,\; Y^{(2)}_{\bar\phi^{34}} = \begin{bmatrix} (e_3\,|b_2+x) \\ (e_4\,|b_2+x) \\ 1+(z_1\,|b_2+x) \\ (z_2\,|b_2+x) \\ (\alpha+x\,|b_2+x) \end{bmatrix} \;,\; Y^{(3)}_{\bar\phi^{34}} = \begin{bmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ 1+(z_1\,|b_3+x) \\ (z_2\,|b_3+x) \\ (\alpha+x\,|b_3+x) \end{bmatrix} \tag{218}$$

$$Y^{(1)}_{\bar\phi^{5..8}} = \begin{bmatrix} (e_1\,|b_1+x) \\ (e_2\,|b_1+x) \\ (z_1\,|b_1+x) \\ 1+(z_2\,|b_1+x) \\ (\alpha+x\,|b_1+x) \end{bmatrix} \;,\; Y^{(2)}_{\bar\phi^{5..8}} = \begin{bmatrix} (e_3\,|b_2+x) \\ (e_4\,|b_2+x) \\ (z_1\,|b_2+x) \\ 1+(z_2\,|b_2+x) \\ (\alpha+x\,|b_2+x) \end{bmatrix} \;,\; Y^{(3)}_{\bar\phi^{5..8}} = \begin{bmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ (z_1\,|b_3+x) \\ 1+(z_2\,|b_3+x) \\ (\alpha+x\,|b_3+x) \end{bmatrix} \tag{219}$$

## 6.2 Duality ?

Looking at the expessions of the projectors for "‘spinorials"’ representations and vectorials representations, we can see that they are very similar. Imposing specific phases to discriminate the different representations of a $B^{(i)}_{pqrs}$ sector we can find the kind a duality between spinors and vectors as in paper ([11]).
If we take $S^{(i)}(\mathbf{rep})_{gg}$ to be the number of representations **rep** of the gauge group $gg$ arising from the sectors $B^{(i)}_{pqrs}$in one model (one set of GGSO phases) and $V^{(i)(osc)}_{gg}$ to be the number of vectorial representations of the gauge group $gg$ arising from the $osc|R >^{(i)}_{pqrs}$ states in the sectors $B^{(i)}_{pqrs}+x$, then we have the following dualities :

$$\begin{array}{lcl} S^{(i)}(\mathbf{4},\mathbf{2},\mathbf{1})_{SO(6)_{obs}\times SO(4)_{obs}} + S^{(i)}(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})_{SO(6)_{obs}\times SO(4)_{obs}} & \longleftrightarrow & V^{(i)(\bar\psi_{123})}_{SO(6)_{obs}} \\ S^{(i)}(\mathbf{4},\mathbf{1},\mathbf{2})_{SO(6)_{obs}\times SO(4)_{obs}} + S^{(i)}(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{SO(6)_{obs}\times SO(4)_{obs}} & \longleftrightarrow & V^{(i)(\bar\psi_{45})}_{SO(4)_{obs}} \end{array}$$

for $i = 1,2,3$. More explicitly, these dualities come from the formulas :

$$\begin{aligned} S^{(i)}(\mathbf{4},\mathbf{2},\mathbf{1})_{...} + S^{(i)}(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})_{...} &= \sum_{pqrs} \frac{1}{2}\left(1 - c\binom{\alpha+x}{B^{(i)}_{pqrs}}\right)\cdot P^{(i)}_{pqrs} \\ V^{(i)(\bar\psi_{123})}_{SO(6)_{obs}} &= \sum_{pqrs} P^{(i)(\bar\psi_{123})}_{pqrs} \\ S^{(i)}(\mathbf{4},\mathbf{1},\mathbf{2})_{...} + S^{(i)}(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{...} &= \sum_{pqrs} \frac{1}{2}\left(1 + c\binom{\alpha+x}{B^{(i)}_{pqrs}}\right)\cdot P^{(i)}_{pqrs} \\ V^{(i)(\bar\psi_{45})}_{SO(4)_{obs}} &= \sum_{pqrs} P^{(i)(\bar\psi_{45})}_{pqrs} \end{aligned} \tag{220}$$

Then we have similarly :

$$S^{(i)}(\mathbf{4},\mathbf{2},\mathbf{1})_{SO(6)_{obs}\times SO(4)_1}+S^{(i)}(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})_{SO(6)_{obs}\times SO(4)_1} \longleftrightarrow V^{(i)(\bar\psi_{123})}_{SO(6)_{obs}}$$
$$S^{(i)}(\mathbf{4},\mathbf{1},\mathbf{2})_{SO(6)_{obs}\times SO(4)_1}+S^{(i)}(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{SO(6)_{obs}\times SO(4)_1} \longleftrightarrow V^{(i)(\bar\phi_{12})}_{SO(4)_1}$$

for $i=4,5,6$.

$$S^{(i)}(\mathbf{4},\mathbf{2},\mathbf{1})_{SO(6)_{obs}\times SO(4)_2}+S^{(i)}(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})_{SO(6)_{obs}\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\psi_{123})}_{SO(6)_{obs}}$$
$$S^{(i)}(\mathbf{4},\mathbf{1},\mathbf{2})_{SO(6)_{obs}\times SO(4)_2}+S^{(i)}(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{SO(6)_{obs}\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\phi_{34})}_{SO(4)_2}$$

for $i=4',5',6'$, with $B^{(4',5',6')}_{pqrs}=B^{(4,5,6)}_{pqrs}+z_1$.

$$S^{(i)}((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))_{SO(4)_{obs}\times SO(4)_1} \longleftrightarrow V^{(i)(\bar\psi_{123})}_{SO(6)_{obs}}$$
$$S^{(i)}((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}))_{SO(4)_{obs}\times SO(4)_1} \longleftrightarrow V^{(i)(\bar\psi_{45})}_{SO(4)_{obs}}$$
$$S^{(i)}((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))_{SO(4)_{obs}\times SO(4)_1} \longleftrightarrow V^{(i)(\bar\phi_{12})}_{SO(4)_1}$$
$$S^{(i)}((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))_{SO(4)_{obs}\times SO(4)_1} \longleftrightarrow V^{(i)(\bar\phi_{34})}_{SO(4)_2}$$

for $i=7,8,9$.

$$S^{(i)}((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))_{SO(4)_{obs}\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\psi_{123})}_{SO(6)_{obs}}$$
$$S^{(i)}((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}))_{SO(4)_{obs}\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\psi_{45})}_{SO(4)_{obs}}$$
$$S^{(i)}((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))_{SO(4)_{obs}\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\phi_{34})}_{SO(4)_2}$$
$$S^{(i)}((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))_{SO(4)_{obs}\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\phi_{12})}_{SO(4)_1}$$

for $i=7',8',9'$, with $B^{(7',8',9')}_{pqrs}=B^{(7,8,9)}_{pqrs}+z_1$.

$$S^{(i)}((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))_{SO(4)_1\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\psi_{123})}_{SO(6)_{obs}}$$
$$S^{(i)}((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}))_{SO(4)_1\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\phi_{12})}_{SO(4)_1}$$
$$S^{(i)}((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))_{SO(4)_1\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\phi_{34})}_{SO(4)_2}$$
$$S^{(i)}((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))_{SO(4)_1\times SO(4)_2} \longleftrightarrow V^{(i)(\bar\psi_{45})}_{SO(4)_{obs}}$$

for $i=10,11,12$.

$$S^{(i)}(\mathbf{8})_{SO(8)_{hid}} \longleftrightarrow V^{(i)(\bar\phi_{5..8})}_{SO(8)_{hid}}$$
$$S^{(i)}(\bar{\mathbf{8}})_{SO(8)_{hid}} \longleftrightarrow V^{(i)(\bar\psi_{123})}_{SO(6)_{obs}}$$

for $i=13,14,15$.

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