A novel, (\(\alpha'\)-)non-perturbative, world-sheet renormalization-group and four-dimensional string solutions of cosmological interest

N. E. Mavromatos

King’s College London, Dept. of Physics

Liverpool U., April 2007
• String Cosmology Basics: Stringy $\sigma$-models in graviton, dilaton, ... cosmological (time-dependent) backgrounds

• A Novel world-sheet Renormalization-Group (NRG) framework: fixed cut-off, floating control parameters.

• String Cosmologies as fixed points of the NRG flow and cosmological evolution: linearly expanding Universes and non-trivial dilatons as trivial exactly marginal solutions of the NRG flow. Non-trivial Dilatons in Minkowski space times as non-trivial fixed points of NRG flows. Correspondence with Infrared fixed points of Wilsonian flows. Exit phase of a linearly expanding Universe?

• WHAT CONSEQUENCES?

$D = 4$ as a new critical space-time dimension of string theory?

Relaxation Dark Energy/Dilaton Quintessence

Dilaton corrections to Boltzmann Equation for thermal relic abundances, Modifications on constraints for Cosmologically appealing particle physics models (e.g. supersymmetric, novel astro-particle phenomenology?)
String Cosmology: Basics

Stringy $\sigma$-models propagating in dilaton $\phi$ and graviton $G_{\mu\nu}$ backgrounds, depending only on target time $X^0 \iff$ Homogeneous Cosmological backgrounds as a first approximation:

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma} \left\{ \gamma^{ab} G_{\mu\nu}(X^0) \partial_a X^\mu \partial_b X^\nu + \alpha' R(2) \phi(X^0) \right\},$$

Background fields as interaction couplings in a world-sheet theory

$$S = S^* + \int_{\Sigma} g^i V_i, \quad g^i = \{ G_{\mu\nu}(X^0), \phi(X^0), \ldots \}$$

Standard two-dimensional field theory renormalization-group analysis, implies renormalized couplings $g_R(\mu), \mu$ a world-sheet RG scale $\iff \beta$-functions,

$$\beta^i \equiv \frac{dg_R^i(\mu)}{d\ln \mu}$$

NB1: Taking into account diffeomorphism invariance of target space: $\tilde{\beta}^i = \text{Weyl anomaly coefficients}, \delta g_R^i = \text{infinitesimal target-space diffeomorphism transformations (passive) of field } g_R^i$, e.g. for $G_{\mu\nu}$:

$$\delta G_{\mu\nu} = \nabla_{(\mu} W_{\nu)} (G, \phi).$$
Local world-sheet conformal invariance requires: $\tilde{\beta}^i = 0$. To first order in $\alpha'$ (Regge slope):

$$
\tilde{\beta}^G_{\mu\nu} = 0 = R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi, \quad \tilde{\beta}^\Phi = 0 = \frac{D - 26}{6} - \frac{1}{2} \nabla^2 \phi + \partial_\rho \phi \partial^\rho \phi.
$$

Equivalent to target-space-time equations of motion from an effective action

$$
\frac{\delta I[g^i]}{\delta g^i} = 0.
$$

Frame choice: (I) $\sigma$-model frame $G_{\mu\nu}$, $\phi$: Non-canonical scalar curvature term in effective action

$$
\int \sqrt{G} e^{-2\phi} R(G) + \ldots
$$

(II) Einstein frame: canonicallly normalized scalar curvature term in effective action

$$
\int \sqrt{g_E} R(g_E) + \ldots \text{ with: } ds^2 = g^{E}_{\mu\nu} dx^\mu dx^\nu = -dt_E^2 + a^2(t_E) d\vec{x}^2, \quad \vec{x} = \{x^1, \ldots x^{D-1}\},
$$

$$
g_{\mu\nu} = e^{-4\phi/(D-2)} G_{\mu\nu}, \quad dt_E = e^{-2\phi/(D-2)} \sqrt{|G_{00}|} dX^0.
$$

**NB: Which is the physical metric?** Depends how you measure...e.g. one may consider matter parts of effective action normalized, then coupling with dilaton will determine the physical metric as the one in which the kinetic terms of matter are canonically normalized.

**Standard Cosmology based on normalized Einstein term (no Brans-Dicke):** Einstein frame the “physical one”: c.f. pre-Big Bang Cosmologies etc. (Veneziano, Gasperini, Piazza...).
Four-Dimensional Effective Cosmologies

Effective 4-d action (after appropriate compactification, freezing out of moduli fields etc.) with matter $I_m$ and radiation in $\sigma$-model frame to $O(\alpha')$

\[ S^{(4)} = \frac{1}{2\alpha'} \int d^4x \sqrt{-G} \left[ e^{-\Psi(\phi)} R(G) + Z(\phi)(\nabla \phi)^2 + 2\alpha' V(\phi) \ldots \right] - \frac{1}{16\pi} \int d^4x \sqrt{G} \frac{1}{\alpha(\phi)} F_{\mu\nu}^2 - I_m(\phi, G, \text{matter}) , \]

including string loops (higher-world-sheet genera),

\[ e^{\Psi(\phi)} = c_0 e^{-2\phi} + c_1 + c_2 e^{2\phi} + \ldots , \]

\[ Z(\phi) = 4 + \ldots , \]

\[ V(\phi) = \text{dilaton potential generated by string loops}. \]

Cosmological Equations (in Einstein frame) for Robertson-Walker Universes:

\[ \dot{\rho}_\phi = (\dot{\phi})^2 + V(\phi)/2 , \quad p_\phi = (\dot{\phi})^2 - V(\phi)/2 , \quad H \equiv \frac{\dot{a}}{a} , \quad a = \text{scale factor} , \quad \dot{A} \equiv dA/dt_E \]

\[ 3H^2 = \rho_m + \rho_\phi , \]

\[ 2 \frac{dH}{dt_E} = -\rho_m - \rho_\phi - p_m - p_\phi , \quad i = 1, 2, 3 , \]

\[ \frac{d^2\phi}{dt_E^2} + 3H \frac{d\phi}{dt_E} + \frac{1}{4} \frac{\partial V}{\partial \phi} + \frac{1}{2} (\rho_m - 3p_m) = 0 , \]

Matter Conservation equations:

\[ \dot{\rho}_m + 3H(\rho_m + p_m) - \dot{\phi}(\rho_m - 3p_m) = 0 \]

$O(\alpha')$ sufficient for late eras, what about higher orders in $\alpha'$ appropriate for earlier eras?

Need for non perturbative configurations... $\Rightarrow$ Novel RG flow on world-sheet...
Important: Absence of horizons in (perturbative) string theory

Important

Perturbative strings (defined through $\sigma$-models) must have well-defined (gauge invariant) Scattering amplitudes ($S$-matrix) $\Rightarrow$ no cosmic horizons ...

$$\delta \sim \int_{t_0}^{t_{\text{End}}} \frac{cdt'}{a(t')} \to \infty \quad \text{if No horizons}$$

NB: Dilaton and moduli potentials in effective actions, constrained by supersymmetry breaking etc. have such form so as to guarantee absence of cosmic horizons and thus well-defined asymptotic states ...

Absence-of-horizons condition must always be taken into account asymptotically in cosmic time ($t \to \infty$)...
Linear Dilaton as exact (in $\alpha'$) solution

(Antoniadis, Bachas, Ellis Nanopoulos 1989)

$$\phi(X^0) = -QX^0, \quad Q^2 = \frac{26 - D}{6}, \quad G_{\mu\nu} = \eta_{\mu\nu}$$

exact (to all orders in $\alpha'$) solution

In EINSTEIN FRAME:

Linearly expanding scale factor

$$a(t_E) \sim t_E,$$

Logarithmic dilaton

$$\phi \sim -\ln(t_E).$$

To interpret expansion define how you measure, e.g. proton decay (Olive et al.) shows real expansion.

How is the Universe exiting from this phase into a Minkowski space time?

Attempt to answer by means of a Novel approach to (world-sheet) Renormalization (Alexandre, Ellis, NM, JHEP0612, 071 (2006) [hep-th/0610072])

Associate linear dilaton with some trivial (Gaussian) fixed point of a flow and the exit (Minkowski space time) phase with a non-trivial (infrared on the world-sheet) fixed point.
Novel Renormalization-Group (NRG) Flow

General Field Theory (Alexandre, Polonyi)

- Fix the ultraviolet cutoff of a field theory (e.g. at Planck (or string) scale)
- Allow for a given physical parameter to “run”, with the cutoff fixed, thus controlling the quantum fluctuations of the theory (Control Parameter (CP)).
- Write down appropriate Flow equations w.r.t. CP and look for fixed point solutions, i.e. solutions that do not depend on this parameter.
- Compare flows with the corresponding Wilsonian RG. Results are usually non perturbative and one obtains differences beyond a certain order in quantum loops (e.g. QED applications yield differences in running of electric charge beyond one loop (Alexandre, Polonyi, Seiler)).

Application to World-sheet (2-d) Renormalization (Alexandre, Ellis, NM)

Use size of $\alpha'$ as a CP, in a sense to be defined below, and thus arrive at fixed point solutions for dilaton and graviton configurations valid to all orders in $\alpha'$. Keep the world-sheet cutoff fixed while the CP runs.
Controlling the Amplitudes of Quantum Fluctuations in stringy cosmology $\sigma$-models:

Consider a spherical world sheet with a curvature scalar $R^{(2)}$. “Bare” $\sigma$-model action:

$$S = \frac{1}{4\pi} \int d^2 \xi \sqrt{\gamma} \left\{ \lambda \gamma^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + R^{(2)} \phi_b (X^0) \right\},$$

$\lambda$ interpolates for $1/\alpha' \implies$ can be used to control the magnitude of the quantum fluctuations, which are proportional to $\alpha'$. It parametrizes the quantum theory described by the effective action, i.e., the proper-graph-generating functional $\Gamma_\lambda$.

The range of values for $\lambda$ is $[1/\alpha'; \infty[$:

- $\lambda \to \infty$ corresponds to $\alpha' \to 0$ and therefore to a classical theory: the kinetic term dominates over the bare dilaton $\phi_B$ term, and the theory is free;

- $\lambda \to 1/\alpha'$ generates to the full quantum theory: the interactions arising from $\phi_B$ are gradually switched on as $\lambda$ decreases from $\infty$ to $1/\alpha'$.

We seek a $\lambda$-independent configuration of the bosonic string, which therefore, by definition, is non-perturbative since it is independent of the strength of $\alpha'$. 

Liverpool U., April 2007
World-sheet NRG Flow: Summary

Bare action for the bosonic string on a Euclidean world sheet, in terms of microscopic fields $\tilde{X}^\mu$:

$$S_B = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \left\{ \gamma^{ab} \lambda \eta_{\mu\nu} \partial_a \tilde{X}^\mu \partial_b \tilde{X}^\nu + R^{(2)} \phi_B (\tilde{X}^0) \right\} ,$$

Add the source term $S_S = \int d^2\xi \sqrt{R^{(2)}} \eta_{\mu\nu} V^\mu \tilde{X}^\nu$, to define classical fields.

NRG flow equation for Effective action $\Gamma$:

$$\frac{d\Gamma}{d\lambda} = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \gamma^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \eta_{\mu\nu} \frac{\eta_{\mu\nu}}{4\pi} \text{Tr} \left\{ \gamma^{ab} \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \xi^b} \left( \frac{\delta^2 \Gamma}{\delta X^\mu (\xi) \delta X^\nu (\xi)} \right)^{-1} \right\} .$$

Assume the following functional dependence:

$$\Gamma = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \left\{ \gamma^{ab} \left( \kappa \lambda (X^0) \partial_a X^0 \partial_b X^0 + \tau \lambda (X^0) \partial_a X^k \partial_b X^k \right) + R^{(2)} \phi_\lambda (X^0) \right\} .$$

Evolution equation for $\phi$:

$$\frac{d\phi}{d\lambda} = - \frac{\Lambda^2}{2R^{(2)}} \left( \frac{1}{\kappa} + \frac{D - 1}{\tau} \right) + \frac{\phi''}{4\kappa^2} \ln \left( 1 + \frac{2\Lambda^2 \kappa}{R^{(2)} \phi''} \right) .$$
SOME TECHNICAL DETAILS
Derivation of World-sheet NRG Flow

Bare action for the bosonic string on a Euclidean world sheet, in terms of microscopic fields $\tilde{X}^\mu$:

$$S_B = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \left\{ \gamma^{ab} \lambda \eta_{\mu\nu} \partial_a \tilde{X}^\mu \partial_b \tilde{X}^\nu + R^{(2)} \phi_B (\tilde{X}^0) \right\},$$

Add the source term $S_S = \int d^2\xi \sqrt{\gamma} R^{(2)} \eta_{\mu\nu} V^\mu \tilde{X}^\nu$, to define classical fields.

The partition function $Z$ and the generating functional of the connected graphs $W$ are:

$$Z = \int \mathcal{D}[\tilde{X}] e^{-S_B - S_S} = e^{-W}.$$

Thus, classical fields $X^\mu$: $X^\mu = \frac{1}{Z} \int \mathcal{D}[^\tilde{X}] \tilde{X}^\mu e^{-S_B - S_S} = \frac{1}{Z} \langle \tilde{X}^\mu \rangle$, $\frac{1}{\sqrt{\gamma} \xi} \frac{\delta W}{\delta V_\mu(\xi)} = X^\mu(\xi)$. Then

$$\frac{1}{\sqrt{\gamma} \xi} \frac{\delta^2 W}{\delta V_\mu(\xi) \delta V_\nu(\xi)} = X_\nu^\prime(\xi) X^\mu(\xi) - \frac{\langle \tilde{X}^\nu(\xi) \tilde{X}^\mu(\xi) \rangle}{Z}.$$

Legendre transform $\Gamma$ of $W$, i.e. the proper-graph generating functional: $\Gamma = W - \int d^2\xi \sqrt{\gamma} R^{(2)} V^\mu X_\mu$, with its functional derivatives:

$$\frac{1}{\sqrt{\gamma} \xi} \frac{\delta \Gamma}{\delta X^\mu(\xi)} = -V_\mu(\xi),$$

$$\frac{1}{\sqrt{\gamma} \xi} \frac{\delta^2 \Gamma}{\delta X_\nu(\xi) \delta X^\mu(\xi)} = - \left( \frac{\delta^2 W}{\delta V_\nu(\xi) \delta V_\mu(\xi)} \right)^{-1}.$$
The evolution of $W$ with $\lambda$ is given by:

$$
\dot{W} = \frac{1}{4\pi Z} \int d^2 \xi \sqrt{\gamma} \gamma^{ab} \eta_{\mu \nu} \left\langle \partial_a \dot{X}^\mu \partial_b \dot{X}^\nu \right\rangle = \frac{\eta_{\mu \nu}}{4\pi Z} \text{Tr} \left\{ \gamma^{ab} \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \zeta^b} \left\langle \dot{X}^\mu(\xi) \dot{X}^\nu(\zeta) \right\rangle \right\},
$$

where a dot denotes a derivative with respect to $\lambda$, and the trace is defined by

$$
\text{Tr}\{\cdots\} = \int d^2 \xi d^2 \zeta \sqrt{\gamma_\xi \gamma_\zeta} \{\cdots\} \delta^2 (\xi - \zeta).
$$

Evolution of $\Gamma$ is obtained by noting that the independent variables of $\Gamma$ are $X^\mu$ and $\lambda$, and that

$$
\dot{\Gamma} = \dot{W} + \int d^2 \xi \frac{\partial W}{\partial \dot{V}_\mu} \dot{V}_\mu - \int d^2 \xi \sqrt{\gamma} R_{\mu \nu} X^\mu = \dot{W}.
$$

Finally, evolution equation for $\Gamma$:

$$
\dot{\Gamma} = \frac{1}{4\pi} \int d^2 \xi \sqrt{\gamma} \gamma^{ab} \eta_{\mu \nu} \partial_a X^\mu \partial_b X^\nu + \frac{\eta_{\mu \nu}}{4\pi} \text{Tr} \left\{ \gamma^{ab} \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \zeta^b} \left( \frac{\delta^2 \Gamma}{\delta X^\mu(\xi) \delta X^\nu(\xi)} \right)^{-1} \right\}.
$$
Assume the following functional dependence:

\[
\Gamma = \frac{1}{4\pi} \int d^2 \xi \sqrt{\gamma} \left\{ \gamma^{ab} \left( \kappa \lambda (X^0) \partial_a X^0 \partial_b X^0 + \tau \lambda (X^0) \partial_a X^k \partial_b X^k \right) + R^{(2)} \phi \lambda (X^0) \right\}.
\]

With the metric \( \gamma^{ab} = \delta^{ab} \) and a constant configuration \( X^0(\xi) = x^0 \), we have:

\[
\frac{\delta^2 \Gamma}{\delta X^0(\xi) \delta X^0(\xi)} = -\frac{\kappa}{2\pi} \Delta \delta^2 (\xi - \zeta) + \frac{R^{(2)} \phi''}{4\pi} \delta^2 (\xi - \zeta),
\]

\[
\frac{\delta^2 \Gamma}{\delta X^j(\zeta) \delta X^k(\xi)} = -\frac{\tau}{2\pi} \Delta \delta^2 (\xi - \zeta) \delta_{jk},
\]

where a prime denotes a derivative with respect to \( x_0 \). In Fourier components,

\[
\frac{\delta^2 \Gamma}{\delta X^0(p) \delta X^0(q)} = \frac{1}{4\pi} \left( 2\kappa p^2 + R^{(2)} \phi'' \right) \delta^2 (p + q)
\]

\[
\frac{\delta^2 \Gamma}{\delta X^j(p) \delta X^k(q)} = \frac{\tau p^2}{2\pi} \delta^2 (p + q) \delta_{jk}.
\]
Derivation of World-sheet NRG Flow

Area of a sphere with curvature scalar $R^{(2)}$ is $8\pi/R^{(2)}$; with the constant configuration $X^0 = x_0$: $\Gamma = 2\phi \lambda(x_0)$, and the trace appearing in flow equation is

$$
\frac{1}{4\pi} \text{Tr}\{\partial \phi (\delta^2 \Gamma)^{-1}\} = -\int \frac{d^2 p}{(2\pi)^2} \left( \frac{p^2}{2\kappa p^2 + R^{(2)} \phi''} + \frac{D-1}{2\tau} \right) \frac{8\pi}{R^{(2)}}
$$

$$
= -\frac{\Lambda^2}{R^{(2)}} \left( \frac{1}{\kappa} + \frac{D-1}{\tau} \right) + \frac{\phi''}{2\kappa^2} \ln \left( 1 + \frac{2\Lambda^2 \kappa}{R^{(2)} \phi''} \right).
$$

(we used that $\delta^2(p = 0) = $ world-sheet area $= \frac{8\pi}{R^{(2)}}$).

The evolution equation for $\phi$ is finally obtained:

$$
\dot{\phi} = -\frac{\Lambda^2}{2R^{(2)}} \left( \frac{1}{\kappa} + \frac{D-1}{\tau} \right) + \frac{\phi''}{4\kappa^2} \ln \left( 1 + \frac{2\Lambda^2 \kappa}{R^{(2)} \phi''} \right).
$$

Look for fixed point solutions, i.e. $\lambda$-independent solutions, consistent with Conformal Invariance.

Absence of quadratic $\Lambda^2$ divergences: $\kappa = -\tau/(D - 1)$

Redefine spatial coordinates $X^k \rightarrow X^k/\sqrt{D - 1}$

$G_{\mu\nu} = \kappa(X^0)\eta_{\mu\nu}$, i.e. $\sigma$-model metric conformally flat.

NB: Linear-$X^0$ Dilaton, $\phi = Q X^0$, $\kappa = -\tau = 1$ configuration: Trivial $\lambda$-flow fixed point, exactly $\lambda$-marginal configuration! Consistent with its $\alpha'$-non-perturbative conformal-invariant nature.
Look for fixed point solutions, i.e. $\lambda$-independent solutions, consistent with Conformal Invariance.

Absence of quadratic $\Lambda^2$ divergences: $\implies \kappa = -\tau/(D-1)$  
Redefine spatial coordinates $X^k \to X^k/\sqrt{D-1} \implies G_{\mu\nu} = \kappa(X^0)\eta_{\mu\nu}$, i.e. $\sigma$-model metric conformally flat.

(I): Linear-$X^0$ Dilaton, $\phi = QX^0$, $\kappa = -\tau = 1 \implies G_{\mu\nu} = \eta_{\mu\nu}$ configuration: Trivial $\lambda$-flow fixed point, exactly $\lambda$-marginal configuration! Consistent with $\alpha'$-non-perturbative nature.

(II): Non-trivial fixed point of $\lambda$-flow (after the redefinition $X^j \to cX^j$, $j$ target spatial index):

$$\phi(X^0) = \phi_0 \ln(X^0), \quad G_{\mu\nu}(X^0) = \frac{\alpha' A}{(X^0)^2} \eta_{\mu\nu}.$$  

Argue that it satisfies conformal invariance at a non-perturbative level, for every $A > 0$.

NB Proof: Exploit the fact, well-known in string theory, that at higher orders in $\alpha'$ the $\beta$ functions are not fixed uniquely, but can be changed by making local field redefinitions: $G_{\mu\nu} \to \tilde{G}_{\mu\nu}$ and $\phi \to \tilde{\phi}$, which leave the (perturbative) string S-matrix amplitudes invariant. This possibility enables us to maintain conformal invariance to all orders in $\alpha'$. (Weyl anomaly coeff. homogenous in $X^0$, $\tilde{\beta}_{\mu\nu}^G \sim \frac{1}{(X^0)^2}$, $\tilde{\beta}^\Phi \sim \text{const.}$ to all orders in $\alpha'$).
Non-trivial $\lambda$-flow fixed points & Conformal Invariance

Consider a configuration with $\kappa(X^0) = F\phi''(X^0)$, $F=$constant.

The dilaton evolution equation reads: $\kappa \dot{\phi} = -\frac{\Lambda^2}{2R(2)} (1 + (D - 1) \frac{\kappa}{\tau}) + \frac{1}{4F} \ln \left( 1 + \frac{2\Lambda^2 F}{R(2)} \right)$ \implies a non-trivial $\lambda$-independent solution $\dot{\phi} = 0$ if:

$$\frac{\kappa}{\tau} = -\frac{1}{D - 1} + \frac{R^{(2)}}{2(D - 1)\Lambda^2 F} \ln \left( 1 + \frac{2\Lambda^2 F}{R(2)} \right) = -c^2,$$

where we have taken the negative sign corresponding to Minkowski signature, appropriate for large cut-off $\Lambda$, in which case the ratio $\kappa/\tau$ is necessarily negative, and $c$ is a positive constant.

After the redefinition $X^j \rightarrow cX^j$, of the space coordinates of the string, the non-trivial $\lambda$-independent solution is such that

$$G_{\mu\nu}(X^0) \propto \phi''(X^0)\eta_{\mu\nu}.$$
Weyl-Invariance Conditions and the new $\lambda$-fixed point

To first order in $\alpha'$, the beta functions are:

$$\beta_{\mu\nu}^g = R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi + \frac{\alpha'}{2} R_{\mu\lambda\rho\sigma} R_\nu^{\lambda\rho\sigma} + O(\alpha')^2,$$

$$\beta^\phi = \frac{D-26}{6\alpha'} - \frac{1}{2} \nabla^2 \phi + \partial^\rho \phi \partial_\rho \phi + \frac{\alpha'}{16} R_{\mu\rho\nu\sigma} R^\mu_{\rho\nu\sigma} + O(\alpha')^2.$$

Consider a configuration satisfying $\kappa(X^0) \propto \phi''(X^0)$, with the power-law dependence

$$\phi' (X^0) = \phi_0 (X^0)^n, \quad \kappa(X^0) = \kappa_0 (X^0)^{n-1}.$$

We have:

$$\beta_{00}^g = \frac{D-1}{2} \frac{n-1}{(X^0)^2} + (n+1)\phi_0 (X^0)^{n-1} + O(\alpha'),$$

$$\beta_{jk}^g = \delta_{jk} \left( \frac{(n-1)(D-2) - 2}{4(X^0)^2} - \phi_0 (X^0)^{n-1} \right) + O(\alpha'),$$

$$\beta^\phi = \frac{D-26}{6\alpha'} - \frac{\phi_0}{4\kappa_0} \left( n+1 + (n-1)(D-1) \right) + \frac{\phi_0^2}{\kappa_0^2} (X^0)^{n+1} + O(\alpha').$$
NB: For \( n = -1 \), each beta function is homogeneous:

\[
\begin{align*}
\beta_{00}^g &= -\frac{D - 1}{(X^0)^2} + \mathcal{O}(\alpha'), \\
\beta_{jk}^g &= \delta_{jk} \frac{D - 1 + 2\phi_0}{(X^0)^2} + \mathcal{O}(\alpha'), \\
\beta^\phi &= \frac{D - 26}{6\alpha'} + \phi_0 \frac{D - 1 + 2\phi_0}{2\kappa_0} + \mathcal{O}(\alpha').
\end{align*}
\]

NB: Impossible for \( \beta_{00}^g \) to vanish.

But, important remark: for the specific choice \( n = -1 \), the next order in \( \alpha' \) of each beta function is homogeneous with the lowest-order term:

\[
\begin{align*}
R_{0\mu\nu\rho} R_0^{\mu\nu\rho} &= \frac{3(D - 1)}{\kappa_0 (X^0)^2}, \\
R_{j\mu\nu\rho} R_k^{\mu\nu\rho} &= -\delta_{jk} \frac{2D - 1}{\kappa_0 (X^0)^2}, \\
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} &= \frac{2(D^2 - 1)}{\kappa_0^2}.
\end{align*}
\]

This is actually valid to all orders in \( \alpha' \) (at each order, Weyl anomaly coefficients of the dilaton and graviton backgrounds consist of products of appropriate powers of Riemann tensors, dilaton (covariant) derivatives, and the necessary contractions by contravariant metric tensors \( G^{\mu\nu} \propto \phi''(X^0) \eta^{\mu\nu} \)).
We then have,

\[
\begin{align*}
\beta^g_{00} &= \frac{1}{(X^0)^2} \sum_{m=0}^{\infty} \xi_m \left( \frac{\alpha'}{\kappa_0} \right)^m, \\
\beta^g_{jk} &= \frac{\delta_{jk}}{(X^0)^2} \sum_{m=0}^{\infty} \zeta_m \left( \frac{\alpha'}{\kappa_0} \right)^m, \\
\beta^\phi &= \frac{1}{\alpha'} \sum_{m=0}^{\infty} \eta_m \left( \frac{\alpha'}{\kappa_0} \right)^m,
\end{align*}
\]

where \(\xi_m, \zeta_m, \eta_m\) are \(\alpha'\)-independent coefficients.

**NB:** \(\kappa_0\) same order as \(\alpha'\), \(\Rightarrow\) expansion in \(\alpha'\) no longer valid.

Can argue that the configuration \(\phi(X^0) = \phi_0 \ln(X^0), \quad \kappa(X^0) = \frac{\alpha' A}{(X^0)^2}\) satisfies conformal invariance at a non-perturbative level.

**NB Proof:** Exploit the fact, well-known in string theory, that at higher orders in \(\alpha'\) the \(\beta\) functions are not fixed uniquely, but can be changed by making local field redefinitions: \(G_{\mu\nu} \rightarrow \tilde{G}_{\mu\nu}\) and \(\phi \rightarrow \tilde{\phi}\), which leave the (perturbative) string S-matrix amplitudes invariant. This possibility of field redefinition enables us to maintain conformal invariance to all orders in \(\alpha'\).
We illustrate this possibility with an explicit calculation to first non-trivial order in $\alpha'$. 

Keep the target-space metric conformally flat $\Rightarrow$ redefinition of metric such that $\tilde{G}_{\mu\nu} \propto G_{\mu\nu}$:

$$
\tilde{G}_{\mu\nu} = G_{\mu\nu} + \alpha' G_{\mu\nu} \left( b_1 R + b_2 \partial^\rho \phi \partial_\rho \phi + b_3 \nabla^2 \phi \right),
$$

$$
\tilde{\phi} = \phi + \alpha' \left( c_1 R + c_2 \partial^\rho \phi \partial_\rho \phi + c_3 \nabla^2 \phi \right)
$$

$$
G_{\mu\nu} + \frac{G_{\mu\nu}}{A} \left( - b_1 (D - 1)^2 + b_2 \phi_0^2 - b_3 (D - 1) \phi_0 \right) = (1 + B)G_{\mu\nu},
$$

$$
\tilde{\phi} = \phi + \frac{1}{A} \left( - c_1 (D - 1)^2 + c_2 \phi_0^2 - c_3 (D - 1) \phi_0 \right) = \phi + C,
$$

$b_1, b_2, b_3, c_1, c_2, c_3 = \text{constants}$, $R, \nabla$ correspond to the metric $G_{\mu\nu}$ and $B, C$ are constants linear in $b_1, b_2, b_3, c_1, c_2, c_3$.

The new beta functions $\beta^g_{\mu\nu} \rightarrow \tilde{\beta}^g_{\mu\nu}$ and $\beta^\phi \rightarrow \tilde{\beta}^\phi$ are obtained via the appropriate Lie derivatives in theory space, as appropriate to the vector nature of the $\beta^i$ functions in this space:

$$
\tilde{\beta}^g_{\mu\nu} - \beta^g_{\mu\nu} = \int \left( \tilde{G}_{\rho\sigma} - G_{\rho\sigma} \right) \frac{\delta \beta^g_{\mu\nu}}{\delta G_{\rho\sigma}} + \int (\tilde{\phi} - \phi) \frac{\delta \beta^g_{\mu\nu}}{\delta \phi} - \int \beta^g_{\rho\sigma} \frac{\delta (\tilde{G}_{\mu\nu} - G_{\mu\nu})}{\delta G_{\rho\sigma}} - \int \beta^\phi \frac{\delta (\tilde{G}_{\mu\nu} - G_{\mu\nu})}{\delta \phi},
$$

and

$$
\tilde{\beta}^\phi - \beta^\phi = \int \left( \tilde{G}_{\rho\sigma} - G_{\rho\sigma} \right) \frac{\delta \beta^\phi}{\delta G_{\rho\sigma}} + \int (\tilde{\phi} - \phi) \frac{\delta \beta^\phi}{\delta \phi} - \int \beta^g_{\rho\sigma} \frac{\delta (\tilde{\phi} - \phi)}{\delta G_{\rho\sigma}} - \int \beta^\phi \frac{\delta (\tilde{\phi} - \phi)}{\delta \phi}.
$$

Liverpool U., April 2007
After long but straightforward computations, we observe that: for any dimension $D$ and any dilaton amplitude $\phi_0$, one can always find a set of parameters $b_1, b_2, b_3, c_1, c_2, c_3$ such that all the three beta functions $\tilde{\beta}^{g}_{00}, \tilde{\beta}^{g}_{jk}, \tilde{\beta}^{\phi}$ vanish, since the latter are homogeneous in $X^0$.

Proof to higher orders in $\alpha'$ via an inductive argument: If conformal invariance is satisfied at order $n$ in $\alpha'$, as in the first-order case worked out above, there are always enough parameters in the redefinitions of the metric and dilaton at the next order, leaving the string configuration unchanged, which enable the beta functions to vanish and hence conformal invariance to be satisfied at the next order $n + 1$ in $\alpha'$. 
Consider an initial bare theory defined on the world sheet of the string, with cut-off $\Lambda$.

The effective theory defined by the action $S_k$ at the scale $k$ is derived by integrating the ultraviolet degrees of freedom from $\Lambda$ to $k$.

Exact renormalization methods: perform this integration infinitesimally, from $k$ to $k - \delta k$, $\implies$ an exact evolution equation for $S_k$ in the limit $\delta k << k$ (Wegner-Houghton method).

**Note that we consider here a sharp cut-off.**

Consider a Euclidean world-sheet metric; for each $k$:

$$S_k = \frac{1}{4\pi\alpha'} \int d^2 \xi \sqrt{\gamma} \left\{ \gamma^{ab} \kappa_k(X^0) \delta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}(X^0) \right\},$$

Integration of ultraviolet degrees of freedom: write dynamical fields $X^\mu = x^\mu + y^\mu$, $x^\mu$ infrared fields with non-vanishing Fourier components for $|p| \leq k - \delta k$, and $y^\mu$ degrees of freedom to be integrated out, with non-vanishing Fourier components for $k - \delta k < |p| \leq k$ only.
Infinitesimal step of the renormalization group transformation:

\[
\exp \left( -S_k - \delta_k [x] + S_k [x] \right) \\
= \exp \left( S_k [x] \right) \int \mathcal{D}[y] \exp \left( -S_k [x + y] \right) \\
= \int \mathcal{D}[y] \exp \left( - \int_k \frac{\delta S_k [x]}{\delta y^\mu (p)} y^\mu (p) - \frac{1}{2} \int_k \int_k \frac{\delta^2 S_k [x]}{\delta y^\mu (p) \delta y^\nu (q)} y^\mu (p) y^\nu (q) \right) , \\
+ \text{higher orders in } \delta k,
\]

where \( \int_k \) = integration over Fourier modes for \( k - \delta k < |p| \leq k \).

Higher-order terms in the expansion of the action are of higher order in \( \delta k \), since each integral involves a new factor of \( \delta k \). The only relevant terms are of first and second order in \( \delta k \), which are at most quadratic in the dynamical variable \( y \), \( \Rightarrow \) Gaussian integral:

\[
\frac{S_k [x] - S_k - \delta_k [x]}{\delta k} = \frac{\text{Tr}_k}{\delta k} \left\{ \frac{\delta S_k [x]}{\delta y^\mu (p)} \left( \frac{\delta^2 S_k [x]}{\delta y^\mu (p) \delta y^\nu (q)} \right)^{-1} \frac{\delta S_k [x]}{\delta y^\nu (q)} \right\} \\
- \frac{\text{Tr}_k}{2 \delta k} \left\{ \ln \left( \frac{\delta^2 S_k [x]}{\delta y^\mu (p) \delta y^\nu (q)} \right) \right\} + \mathcal{O}(\delta k),
\]

where the trace \( \text{Tr}_k \propto \delta k \) is on a shell of thickness \( \delta k \).
For dilaton evolution equation suffices to consider a constant infrared configuration \( x^\mu \) (c.f. use of a sharp cut-off: no singular terms that could arise from the \( \theta \) function, since derivatives of the infrared field vanish).

In the limit: \( \delta k \to 0 \) gives:

\[
R^{(2)} \partial_k \phi_k(x^0) = -k \ln \left( \frac{2\kappa_k(x^0)k^2 + \alpha' R^{(2)} \phi''_k(x^0)}{2\kappa_k(1)k^2 + \alpha' R^{(2)} \phi''_k(1)} \left( \frac{\kappa_k(x^0)}{\kappa_k(1)} \right)^{D-1} \right),
\]

where a prime denotes a derivative with respect to \( X^0 \) and we chose the dilaton to vanish for \( x^0 = 1 \).

This equation is exact and non-perturbative: it provides a resummation in \( \alpha' \), since the quantities in the logarithm are the running, dressed quantities. **World-sheet Wegner-Houghton (wsWH).**
Fixed-Point Solutions of world-sheet Wegner-Houghton Equation

(i) Trivial (linear-dilaton) fixed point:
A linear dilaton/flat metric configuration $\kappa(x^0) = 1$, $\phi(x^0) = Qx^0$ where the constant $Q$ is independent of $k$, satisfies the wsWH evolution equation. Exactly marginal configuration, independent of the Wilsonian scale $k$ (expected, as it does not generate any quantum fluctuations).

(ii) Non-trivial Infrared fixed point:
Look for a similar exact solution of the renormalization-group equation, in which the graviton is: $\kappa(X^0) = \frac{\alpha' A}{(X^0)^2}$ and the dilaton $\phi_k(x^0) = \eta_k \ln(x^0)$, where $\eta_k$ is a function of $k$.

Such a configuration indeed satisfies the wsWH equation, provided $d\eta_k/dk = 2Dk/R^{(2)}$, and hence

$$\phi_k(x^0) = \left(\phi_0 + \frac{Dk^2}{R^{(2)}}\right) \ln(x^0),$$

where $\phi_0$ is the constant of integration. Therefore, we have been able to find an exact solution of the renormalization-group equation wsWH which tends to the non-trivial fixed-point solution of the $\lambda$-flow solution in the infrared limit $k \to 0$, with a vanishing derivative

$$\partial_k \phi_k(x^0) \to 0^+,\quad$$

and thus we conclude that this configuration is a Wilsonian infrared-stable fixed point.
OPEN ISSUES

• Is linear-dilaton fixed point connected through a NRG trajectory to non-trivial fixed point, so that the latter constitutes Exit Phase of linear dilaton expanding Universe?

Space of background space-time fields, over which strings propagate:
\( \{g^i\} = \{\text{graviton} = G_{\mu\nu}, \text{matter}, \ldots\} \), and Dilaton \( \phi(X^0) \), Irreversible Relaxation Flow (RG) between string vacua (equilibrium points) from world-sheet UV to IR fixed points (Irreversibility \( \iff \) Zamolodchikov C-theorem).
The relation between the physical metric in the Einstein frame and the string metric is given by

\[ ds^2 = dt^2 - a^2(t) dx^k dx^k = \kappa(x^0) \exp \left\{ -\frac{4\phi}{D-2} \right\} \left( dx^0 dx^0 - dx^k dx^k \right), \]

where \( a(t) \) is the scale factor of a spatially flat Robertson-Walker-Friedmann Universe, and the \( x^\mu \) are the zero modes of \( X^\mu \).

From the configuration of the non-trivial \( \lambda \)-flow solution, we have

\[ \frac{dt}{dx^0} = \varepsilon \sqrt{|\kappa_0|(x^0)^{-1 \frac{2\phi_0}{D-2}}} \quad \Rightarrow \quad t = T + \sqrt{|\kappa_0|} \frac{(D-2)}{2|\phi_0|} (x^0)^{\frac{2\phi_0}{D-2}}, \]

where \( \varepsilon = \pm 1 \) and \( T \) a constant.

We find then a power law for the evolution of the scale factor:

\[ a(t) = a_0 |t - T|^{\frac{D-2}{2\phi_0}} + 1, \]

which is in general singular as \( t \to T \).

\[ \text{NB: Expanding Universe:} \quad \frac{D-2}{2\phi_0} + 1 > 0. \quad \text{Absence of cosmic horizons:} \quad \phi_0 < 0 \]
In order to have a Minkowski target space, one needs $D - 2 + 2\phi_0 = 0$.

As discussed above, when dealing with the conformal properties of the configuration, the choices of $D$ and $\phi_0$ are free, and lead to the determination of $\kappa_0$. As a consequence, for a given dimension $D$, it is always possible to choose $\phi_0$ so that the target space is static and flat.

It may therefore find an application to the exit phase from the linearly expanding Universe associated with the linear dilaton.

In terms of the Einstein time $t$, the dilaton can be written, up to a constant, as:

$$\phi = -\frac{D - 2}{2} \ln |t - T|.$$ 

We observe that, like the scale factor, the dilaton has a singularity as $t \to T$. It would be interesting to explore the applicability of this configuration to primordial cosmology.

**NB:** The sign of the expression for the dilaton when $D > 2$ ensures that the string coupling $g_s = e^\phi$ is small at large times.
Strings in $D = 4$ Space-time dimensions?

(J. Alexandre & N.M., hep-th/0703171)

Target space Wick rotation

$$X^0 \rightarrow iX^0.$$ 

Appropriate for a well-defined world-sheet path integral, field $X^0(\xi)$ does not have negative norm. Minkowski metric $\Rightarrow$ Euclidean one, since

$$\eta_{\mu\nu} (X^0)^2 \partial_a X^\mu \partial_b X^\nu \rightarrow \frac{\partial_a (iX^0) \partial_b (iX^0)}{(iX^0)^2} - \delta_{jk} \frac{\partial_a X^j \partial_b X^k}{(iX^0)^2} = \frac{\delta_{\mu\nu}}{(X^0)^2} \partial_a X^\mu \partial_b X^\nu.$$ 

Partition function:

$$Z = \frac{\int \mathcal{D}[X^\mu] \exp(-S)}{\int \mathcal{D}[X^\mu]} , \quad \int \mathcal{D}[X^\mu] = V \quad \text{target – space volume.}$$

The analytic continuation implies action shift:

$$S \rightarrow \tilde{S} = S_E + \frac{i\pi}{2} \phi_0 \chi,$$

$$S_E = \frac{1}{4\pi\alpha'} \int d^2 \xi \sqrt{\gamma} \left\{ \gamma^{ab} \frac{A_{\mu\nu}}{(X^0)^2} \partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)} \phi_0 \ln(X^0) \right\} , \quad \chi = \frac{1}{4\pi} \int d^2 \xi \sqrt{\gamma} \gamma^{ab} R^{(2)} = \text{world-sheet Euler characteristic.}$$

For a closed world sheet, without cross-caps, $\chi = 2 - 2g$, where $g$ is the number of handles.
Hence,

$$\mathcal{Z} \rightarrow \tilde{\mathcal{Z}} = \mathcal{Z}_E \exp\{i\pi(1-g)\phi_0\},$$

where $\mathcal{Z}_E$ is the partition function corresponding to the action $S_E$. For $\tilde{\mathcal{Z}}$ to be real, we need the quantization condition

$$(1-g)\phi_0 = n, \quad n = \text{integer} \neq 0, \quad g \neq 1$$

(since $\phi_0 \neq 0$ in our solution). This yields for the dimension $D$

$$D = 2 - \frac{2n}{1-g}, \quad g \neq 1,$$

in order to have a Minkowski target space. For $\phi_0 < 0$, we have $n/(1-g) < 0$.

For world-sheet sphere, $g = 0$, $D = 4, 6, 8, ..., 10, ..., 26$, NB: Minimum number $D = 4$.

If solution valid for higher genera, then for $g \geq 2$, $n > 0 \Rightarrow D = 4$ can still be a valid critical number (appropriate choice of $n > 0$).

It remains to be seen whether consistent ghost-free string theories can be formulated in these new critical dimensions...
A non-constant dilaton at late times, implies, for $D \neq D^*$ ($D^*$=critical number of dimensions), time-dependent contribution to the dark energy sector of the Universe's energy budget (Einstein frame):

\[ V_{DE} \ni \int d^4x \sqrt{-g_E} e^{2\phi}(D - D^*) \]

For both trivial and non-trivial $\lambda$-flow fixed point solutions (in Einstein frame):

\[ \phi \sim -\ln(t_E), \quad \text{Dark Energy density} \sim 1/t_E^2 \]

Non-trivial fixed point solution remains solution under constant shifts $X^0 \to X^0 + c$:

\[ X^0 \ll c : \quad G_{\mu\nu} = \alpha' A \frac{1}{(X^0+c)^2} \eta_{\mu\nu} \to (\alpha' A/c^2) \left( \eta_{\mu\nu} + O(\frac{X^0}{c}) \eta_{\mu\nu} \right) , \]

\[ \phi = \phi_0 \ln(X^0 + c) \to \phi_0 \ln(c) + \frac{\phi_0}{c} X^0 + \ldots \]

i.e. linear dilaton solution is recovered (within the non-trivial solution) for small target times, while non trivial fixed point applies to intermediate and late times of the Universe evolution.

Some phenomenological consequences: dilaton terms & thermal relics... (NB: dilaton in our model does not acquire a stable vev, hence no supermassive non-thermal dark-matter relics...)
Dilaton modifications of Boltzmann Equation for species abundances

Consequences: Dilaton dissipative source Modifications to evolution equation for (relic) Species Abundances (Boltzmann) (Lahanas, NM, Nanopoulos, hep-ph/0608153)

\[
\left( \hat{L}_{\text{conv}} + \hat{L}_{\text{off-shell,dil}} \right) f = C[f],
\]

\[
\frac{\partial f}{\partial t} = \frac{\dot{a}}{a} \frac{|\vec{p}|^2}{E} \frac{\partial f}{\partial E} - \dot{\Phi} \frac{\partial f}{\partial \Phi} + \frac{1}{E} C[f].
\]

t=cosmic time (Einstein frame), \(\dot{\text{dot}}=\) differentiation w.r.t. \(t\), \(f(|\vec{p}|, t, \Phi(t, \rho), g_{\mu\nu}(t, \rho))\) phase-space density of species, \(C[f] = \) collision term,

Final form of Boltzmann equation (in Einstein frame)

\[
\frac{dn}{dt} + 3 \left( \frac{\dot{a}}{a} \right) n - \dot{\Phi} n = \int \frac{d^3p}{E} C[f]
\]

\(n = \int d^3p f\) number density of species, .

NB: time-dependent dilaton effects, for non-trivial \(\lambda\)-flow fixed-point Minkowski solution \(\dot{a} \sim 0, \dot{\Phi} \sim -1/t \Rightarrow n \propto 1/t\) asymptotically (exit phase), ignoring collisions.
Away from (Minkowski) fixed point, rewrite Boltzmann conveniently in terms of a source $\Gamma (= \dot{\phi}):$

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a}n = \Gamma(t)n + \int \frac{d^3p}{E} C[f].$$

Equation for $Y \equiv n/s$, ($s =$ entropy/unit volume assumed constant after inflation)$dY = \frac{m_{\tilde{\chi}}}{x} \langle v\sigma \rangle \left(\frac{45}{\pi} G_N \tilde{g}_{eff}\right)^{-1/2} \left(h + \frac{x}{3} \frac{dh}{dx}\right) (Y^2 - Y_{eq}) - \frac{\Gamma}{Hx} \left(1 + \frac{x}{3h} \frac{dh}{dx}\right) Y.$$

$x \equiv T/m_{\tilde{\chi}}$, concentrate on a particular species $\tilde{\chi}$, of mass $m_{\tilde{\chi}}$: dominant Dark Matter candidate. $h =$entropy degrees of freedom, $\langle v\sigma \rangle =$ thermal avrg of relative velocity $\times$ annihilation cross section,

$$\rho + \Delta \rho \equiv \frac{\pi^2}{30} T^4 \tilde{g}_{eff}, \quad \rho = \frac{\pi^2}{30} T^4 \tilde{g}_{eff}(T), \quad \Delta \rho = \rho_\phi$$

**NB:** only the degrees of freedom involved in $\rho$ are thermal, while $\rho + \Delta \rho$ are involved in evolution

$$H^2 = \frac{8\pi G_N}{3} (\rho + \Delta \rho), \text{ hence } \tilde{g}_{eff} = g_{eff} + \frac{30}{\pi^2} T^{-4} \Delta \rho.$$
Consequences for (Astro) Particle Physics Constraints (cont’d)

Freeze-out point $x_f$ modification:

$$x_f^{-1} = \ln \left[ 0.03824 \ g_s \ \frac{M_{\text{Planck}} \ m_{\tilde{\chi}} \ x_f^{1/2} \langle v\sigma \rangle_f}{\sqrt{g_*}} \right] + \frac{1}{2} \ ln \left( \frac{g_*}{\tilde{g}_*} \right) + \int_{x_f}^{x_{in}} \frac{\Gamma H^{-1}}{x} \ dx \ .$$

To calculate the relic abundance, solve evolution for $Y$ from $x_f$ to present $x_0$, (temperature $T_0 \approx 2.7^0 K$) (Lahanas, NM, Nanopoulos):

$$Y^{-1}(x_0) = Y^{-1}(x_f) + \left( \frac{\pi}{45} \right)^{\frac{1}{2}} m_{\tilde{\chi}} \ M_{\text{Planck}} \ \tilde{g}_*^{-\frac{1}{2}} \ h(x_0) J - \int_{x_0}^{x_f} \frac{\Gamma H^{-1}}{xY} \ dx \ .$$

Approximate analytical treatment $[ \psi(x) \equiv x \ \exp(- \int_{x_0}^{x} \frac{\Gamma H^{-1}}{xY} dx/x)]$

$$(h(x_0)Y(x_0))^{-1} = \left( 1 + \int_{x_0}^{x_f} \frac{\Gamma H^{-1}}{\psi(x)} \ dx \right)^{-1} \left( \frac{\pi}{45} \right)^{\frac{1}{2}} m_{\tilde{\chi}} \ M_{\text{Planck}} \ \tilde{g}_*^{-\frac{1}{2}} \ J$$

Matter density of species $\tilde{\chi}$:

$$f \equiv \left( 1 + \int_{x_0}^{x_f} \frac{\Gamma H^{-1}}{\psi(x)} \ dx \right) = \exp \left( \int_{x_0}^{x_f} \ dx \ \frac{\Gamma H^{-1}}{x} \right)$$

$$\rho_{\tilde{\chi}} = f \left( \frac{4\pi^3}{45} \right)^{1/2} \left( \frac{T_{\tilde{\chi}}}{T_{\gamma}} \right)^3 \ M_{\text{Planck}}^3 \ \sqrt{\tilde{g}_*} \ \frac{\sqrt{\tilde{g}_*}}{J} \ \approx \ f \left( \frac{4\pi^3}{45} \right)^{1/2} \frac{43}{11} \ M_{\text{Planck}}^3 \ \frac{T_{\gamma}^3}{J} \sqrt{\tilde{g}_*} \ .$$

From this calculate relic abundance...
Modified relic abundance

\[ \Omega_\tilde{\chi} h_0^2 = (\Omega_\tilde{\chi} h_0^2)_{\text{no-source}} \times \left( \frac{\tilde{g}^*_s}{g^*_s} \right)^{1/2} \exp \left( \int_{x_0}^{x_f} dx \frac{\Gamma H^{-1}}{x} \right), \]

with \( (\Omega_\tilde{\chi} h_0^2)_{\text{no-source}} = \frac{1.066 \times 10^9 \text{ GeV}^{-1}}{M_{\text{Planck}} \sqrt{g^*_s}} J = \int_{x_0}^{x_f} \langle v\sigma \rangle dx. \)

(a) if \( \Gamma < 0 \) at all times \( \implies \) reduction of the relic density with time: predictions for supersymmetric models may be drastically altered, since parameter space is enlarged, \( \implies \) more room for supersymmetry, probably beyond the reach of LHC (even in case of constrained minimal SUSY standard models with compact parameter spaces of embedding minimal SUGRA)

(b) if \( \Gamma > 0 \) at all times, relic density increases, parameter space shrinks, predictions can be very restrictive, to almost excluding supersymmetry, (especially if prefactor large number.)

(c) For non-Minkowski power-law \( \lambda \)-flow fixed point Universe:

\[ \Gamma H^{-1} \sim -\frac{\phi_0}{1 + \frac{2}{D-2} \phi_0}, \implies \Omega_\tilde{\chi} h_0^2 = (\Omega_\tilde{\chi} h_0^2)_{\text{no-source}} \times \left( \frac{\tilde{g}^*_s}{g^*_s} \right)^{1/2} \left( \frac{x_0}{x_f} \right)^{1 + \frac{2}{D-2} \phi_0} \]

Smooth connection with Minkowski fixed point (ignoring collisions) requires the following temperature

\[ T \sim \frac{x_0}{x_f} - \text{cosmic time relation: } (x_0/x_f) \sim (\frac{1}{T})^{\frac{D-2+2\phi_0}{2\phi_0}} \sim a - \frac{2}{D-2} \implies \text{for } D=4 \text{ we have Wien's law of thermodynamics as a result of cosmic red-shift } a \propto \lambda \sim 1/T, \lambda \text{ photon wavelength.} \]
FURTHER OPEN ISSUES

• Time dependent dilatons should NOT affect nucleosynthesis era delicate balance (c.f. some (non-equilibrium) Brane-Universe models, where such a feature can be achieved).
String theory is a mathematically consistent model of Quantum Gravity. Quantum Gravity is ultimately linked with Cosmology.

Cosmically-Catastrophic initial events, such as Big-Bang, or Collision of Brane Worlds (modern version of strings), may induce Non-Equilibrium Physics $\Rightarrow$ Relaxation Models for Dark Energy: present dark energy remnant of initial non-equilibrium catastrophic event.

Formal description: Non-Critical (non-conformal, Liouville) Strings describing excitations on our brane world: Non-critical strings approach asymptotically (in time) critical (conformal) strings (equilibrium points), e.g. linear-dilaton fixed point. May be our non-trivial fixed point is also an asymptotic point, being conformal. Asymptotically vanishing dark energy, perturbative $S$-matrix (corner stone of critical strings) well-defined...

Consequences:

(i) of Stringy description: Dilaton couplings with matter, time-dependent dilaton source terms in Boltzmann equation for relic abundances

(ii) of Non-critical stringy description: Off-shell Einstein Cosmological equations (relaxation scenaria), relaxing to zero dilaton dark energy, current acceleration, extra source terms in Boltzmann equation $\Rightarrow$ further modifications on relic abundances, sources $\Gamma = \dot{\phi} + O(\tilde{G}, \phi, \ldots)$. 

\[
\Gamma = \dot{\phi} + O(\tilde{G}, \phi, \ldots).
\]
Space of background space-time fields, over which strings propagate:

\[ \{ g^i \} = \{ \text{graviton} = G_{\mu\nu}, \text{matter}, \ldots \}, \text{and Dilaton } \phi(X^0, \rho) \]

\[ \rho = \text{Liouville mode, Irreversible Relaxation Flow (RG) between string vacua (equilibrium points) (Irreversibility } \iff \text{ Zamolodchikov C-theorem).} \]

\( \rho \) is ESSENTIAL in restoring conformal invariance, perturbed by a NON-EQUILIBRIUM PROCESS, e.g. Catastrophic Cosmic Events (Brane World Collision), or space-time foam (microscopic black holes, space-time defects etc.) , ...
Colliding Brane Worlds with Recoil (Ellis, NM, Nanopoulos, Westmuckett (PRD, IJMPA))

Relative recoil velocity $V$ of branes
Adiabatic collisions allow for stringy computation of brane potential to leading order in $V$

Identical branes (same tension) potential is

$$V_{\text{potential}} = \alpha V^4$$

Stringy Excitations on brane world feel central charge deficit

$$Q^2_{\text{initial}} = O(V^4)$$

RELAXATION PROCESS AFTER COLLISION

$$Q(t) = \frac{v^2}{t}$$

EFFECTIVE LIOUVILLE STRINGY COSMOLOGY ON THE BRANE

$\phi, G_{\text{UV}}, Q(t)$

Logarithmic (super)Conformal Field Theory Techniques to compute scaling with cosmic time, PLUS identification of Liouville mode (IRREVERSIBLE local RG scale on world-sheet) with time (Gravanis, Szabo, N.M.)

COLLISION $\leftrightarrow$ NON-EQUILIBRIUM, IRREVERSIBLE
IMPORTANT: Dilaton time dependent effects should not affect nucleosynthesis
In Liouville string (Non-Equilibrium, off-shell) Dark Energy Models, Dilaton Dark Energy may be negligible at NUCLEOSYNTHESIS epoch. Dilaton is not acquiring a stable vev.
Conformal Field Theory (Logarithmic CFT, in brane recoil models) → asymptotic scaling with cosmic time $\sim \frac{1}{t^2}$ (E.Gravanis, N.M.), c.f. our non-trivial $\lambda$-flow fixed point.
NB: Cosmic Time $\iff$ world-sheet Renormalization Group (RG) local Scale (Liouville mode), Irreversible (Zamolodchikov C-theorem)!
The energy density carried by the dilaton and the non-critical terms (dashed-dotted red line), the deceleration (dashed blue line), the Hubble expansion rate (dashed - double dotted grey line) and the derivative of the cosmic scale factor (solid green line) as functions of the redshift in the range $0 < z < 1.6$ are displayed. Their values refer to the left $y$-axis. The ratio $|q|/g_s^2$ is also displayed (dotted brown line) with values on the right vertical axis.
Left panel: The ratio $|q|/g_s^2$ ($q=$ deceleration, $g_s = e^{\Phi}=$ string coupling) as function of the redshift for $z$ ranging from $z = 0.2$ to future values $z = -0.6$. The rapid change near $z \approx 0.16$ signals the passage from deceleration to the acceleration period. Right panel: The values of the string coupling constant plotted versus redshift value in the range $z = 0.0 - 1.0$.

**NB:** The string coupling drives the acceleration of the Universe asymptotically.
LHC SUSY detection prospects
and
non-critical strings

**Left:** In the thin green (grey) stripe the neutralino relic density is within the WMAP3 limits $0.0950 < \Omega_{CDM} h^2 < 0.1117$, for values $A_0 = 0$ and $\tan \beta = 10$, according to the conventional calculation. The dashed double dotted line (in blue) delineates the boundary along which the Higgs mass is equal to $114.0 \text{ GeV}$. The dashed lines (in red) are the $1\sigma$ boundaries for the allowed region by the $g - 2$ muon’s data as shown in the figure. The dotted lines (in red) delineate the same boundaries at the $2\sigma$’s level. In the hatched region $0.0950 > \Omega_{CDM} h^2$, while in the dark (red) region at the bottom the LSP is a stau. **Right:** The same as in left panel, but according to the non-critical-string calculation in which the relic density is reduced.
**Left:** In the very thin green (grey) stripe the neutralino relic density is within the WMAP3 limits $0.0950 < \Omega_{CDM} h^2 < 0.1117$, for values of $A_0 = 0$ and $\tan\beta = 40$ shown in the figure, according to the conventional calculation. The thin dark (purple) region lying above is the same region according to the non-critical-string calculation with the reduction factor for the MSSM inputs shown in the figure. The remaining Higgs and $g - 2$ boundaries are as in figure 44. The hatched dark (cyan) region on the left is excluded by $b \rightarrow s \gamma$ data. **Right:** The same as in left panel, for $A_0 = 0$ and $\tan\beta = 55$. 
Novel world-sheet renormalization-group approach to stringy cosmology $\sigma$-models: flow w.r.t. a control parameter, fixed world-sheet cut-off.

Linearly expanding Universes and logarithmic dilatons characterize trivial fixed points of the new flow.

New non-perturbative (in $\alpha'$) cosmologies as non-trivial marginal points of the new flow w.r.t. the control parameter (or infrared fixed points of a corresponding Wilsonian flow).

Minkowski space-times and dilatons depending logarithmically on Einstein-frame time characterise the non-trivial fixed points. Exit phase from linearly expanding Universe? $D = 4$ as a new critical space-time dimension of string theory? (Upon target-time analytic continuation)

Non-trivial dilaton Modifications to Boltzmann equation, important consequences for Astrophysical (WMAP etc.) constraints on interesting (e.g. supersymmetric) particle physics models... Nucleosynthesis era & time-dependent dilatons? (OPEN ISSUES, non-equilibrium?)

Apply this method to study phase transitions in early Universe, or even to Liouville strings (Alexandre, Ellis, NM, JHEP0703,060 (2007)), with time-like Liouville mode as target time, to study non-equilibrium stringy cosmologies with Dilaton Quintessence/Relaxation-Dark-Energy...