Dynkin diagrams

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Abstract

We present brief notes on the subject of Dynkin diagrams, to accompany those offered by WW at the String Journal Club meeting of 18th Feb 2013.

1 Preliminaries

Dynkin diagrams, as they are used and understood by physicists, provide a useful and elegant pictorial language for talking about simple Lie algebras. The first reference one should approach in order to learn more, or to have by one's side whilst attempting examples, is [1].

1.1 Lie algebras

We'll start by reviewing some basic ideas of the theory of Lie algebras, which we will use throughout. The assumption is that the reader is either familiar with, or knows where to look for, most of the information here, which is provided predominantly for completeness.

Let G be some Lie group (which we take throughout to be simple) and $\mathfrak{g}$ its corresponding Lie algebra, having generators T_A for $A = 1, \ldots, \dim(G)$. The nature of the algebra is encoded in the form of the **structure constants**

$$[T_A, T_B] = f_{AB}{}^C T_C, \tag{1}$$

and we will become increasingly sloppy with the position of the Lie algebra indices when indeed we do use them.

We define the **rank** of $\mathfrak{g}$ as the maximal number of simultaneously diagonalisable generators. In more mathematical parlance this is the dimension of the maximal **Cartan subalgebra** $\mathfrak{h} \subset \mathfrak{g}$, that is the subalgebra of all generators H_i with $i = 1, \ldots, l = \mathrm{rank}(\mathfrak{g})$ satisfying

$$[H_i, H_j] = 0. \tag{2}$$

The rest of the generators, which we denote $E_{\vec{\alpha}}$ are then eigenfunctions of the Cartan generators H_i. That is, they satisfy

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}, \tag{3}$$

where the vector $\vec{\alpha} \in \mathbb{R}^l$ we call a **root**.

In order to talk about physics things like particles, we need to understand the irreducible representations of this algebra, which we can label by their eigenvalues under the action of the Cartan generators H_i. In particular, we label irreps by a **weight** λ as

$$H_i|\lambda\rangle = \lambda_i|\lambda\rangle. \tag{4}$$

This is reminiscent of the story we learned in quantum mechanics courses, where the Lie algebra of interest was $\mathfrak{su}(2)$, which has rank one, and we classified irreducible representations by their eigenvalues under J_3, i.e. their spin.

In order to clear things up a bit, let's be a bit more precise about what we mean by (4). Let's take an n-dimensional representation ρ of the rank-l Lie algebra $\mathfrak{g}$. Then the l Cartan generators H_i can be represented as $n \times n$ diagonal matrices $\rho(H_i)$ with diagonal elements μ_i^a for $a = 1, \ldots, n$. These matrices act naturally on n-vectors, which we take to have the usual basis $\{e_1, \ldots, e_n\}$. In other words, we have

$$H_i = \mu_i^a e_a.$$

Then the l-vector $|\lambda^{(a)}\rangle \in \mathbb{R}^l$ will have components $(\mu_i^1, \ldots, \mu_i^n)$, so that we have

$$H_i|\lambda^{(a)}\rangle = \mu_i^a|\lambda^{(a)}\rangle \equiv \lambda_i|\lambda^{(a)}\rangle. \tag{5}$$

As a quick example, we can look at the 3-dimensional representations of $SU(3)$. The algebra $\mathfrak{su}(3)$ has rank 2, so there are two Cartan generators which we take to be

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then the states $|\lambda\rangle$ are denoted by $\{(1,1), (1,0), (-2,-1)\}$.

We can now use all this to say something about how states in a representation are 'generated'. In particular, we note that

$$H_i E_{\vec{\alpha}}|\lambda\rangle = (\alpha_i + \lambda_i) E_{\vec{\alpha}}|\lambda\rangle,$$

which tells us that, whenever $\alpha_i + \lambda_i$ is non-zero, the state $E_{\vec{\alpha}}|\lambda\rangle$ is proportional to the state $|\lambda + \alpha\rangle$. It is in this manner that we can build a 'representation' (in the physics language) by starting with some highest-weight state and acting with successive $E_{\vec{\alpha}}$ to get other states in a **multiplet**.

Before we move on to introduce Dynkin diagrams, we need some terminology. Given some set of **roots** λ, we define a **positive root** to be one such that its first non-zero element (in the chosen basis) is positive. In the example above, our two positive roots are given by $(1, 1)$ and $(1, 0)$. We then define the **simple roots** as those positive roots which cannot be written as a sum of the other positive roots with positive coefficients. Again, in the example above, both our positive roots are also simple roots.

It is a theorem that the number of simple roots is equal to the rank of the Lie algebra. In particular, the simple roots form a basis of the Cartan subalgebra.

All of these facts have just been thrown at you, and admittedly that is how they are presented in [1]. For those who want a bit more rigour with their mains, there are many good mathematical references on these subjects. I shall mention just one, [2], which contains a fairly readable account, with proofs, of most of the material here presented.

2 Dynkin diagrams

We'll now get on with the meat of these notes, which concern Dynkin diagrams. Although one can represent any simple Lie algebra by looking at the root diagram, which lives in some l-dimensional space, this is neither particularly illuminating nor particularly easy.

Thankfully, there is an easier method introduced by Dynkin which allows one to classify all simple Lie algebras.

First, we introduce a set of **nodes**, whose number is equal to the rank of the Lie algebra $\mathfrak{g}$ we're interested in. Each of these nodes correspond to a particular simple root. The Dynkin diagram encodes the lengths and relative angles of these roots.

It turns out that in any simple Lie algebra, simple roots can only have two lengths, some 'long' and some 'short'. The short roots we represent by a filled-in node whilst the long ones are represented by a hollow node.

We also add in lines to the diagram, which connect the nodes. Given any two simple roots, α_i, α_j, we draw a single line between the corresponding nodes if the angle between α_i and α_j is $120°$, a double line if it is $135°$, and a triple line if it is $150°$. If the angle is $90°$ we put no line, since then the roots are orthogonal.

One can also show that the angles also encode the relative lengths of the roots. Namely, they are the same length is there is a single line, there's a ratio of $\sqrt{2}$ for a double line, and a ratio of $\sqrt{3}$ if there's a triple line.

With all this in place, we can (if we are Dynkin, which we're probably not) go on and classify all the simple (and semi-simple) Lie algebras. Thankfully, we don't have to be Dynkin, because he did that job so well himself, so we can look in a table to see the classification. The complete list is given in, amongst other places, Table 5 of [1]. This table also gives the labelling of the nodes, which will be important in what follows.

2.1 The simply-laced examples

If we look closely at the classification, we can see that the ADE-algebras only have single lines joining the nodes, i.e. all the simple roots have the same length. These are known as the **simply-laced** Lie algebras, and form an especially useful set of examples for physicists interested in phenomenology and GUT model-building. These are the examples we will concentrate on for the remainder of these notes.

3 The Cartan matrix

Another method of presenting the useful information in a Lie algebra is via. the so-called **Cartan matrix** A_{ij}, which has components

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \tag{6}$$

where the α_i here are the simple roots. For the simply-laced examples that we are interested in all of the simple roots have the same length, which we normalise to 2, so that (6) becomes

$$A_{ij} = (\alpha_i, \alpha_j). \tag{7}$$

Remembering our high-school trigonometry, and the rules regarding lengths and angles between simple roots mentioned above, we see that, for the case where $i \neq j$ but there exists a single line between the nodes i and j, we have

$$(\alpha_i, \alpha_j) = |\alpha_i||\alpha_j| \cos(120^\circ) = -1,$$

whereas for $i = j$ we have $(\alpha_i, \alpha_i) = 2$. Hence, for the simply-laced algebras, the Cartan matrix is an $l \times l$ matrix with 2 on the diagonal and a -1 whenever two nodes are joined by a line in the Dynkin diagram.

For example, the Cartan matrix for $\mathfrak{su}(5)$, which has rank 4, is

$$A_{SU(5)} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

4 Some theorems of Dynkin

Now that we've covered the fundamentals of Dynkin diagrams, we'll go on to look at some theorems which help us to understand better the structure of the irreps that we need for physics.

First of all we want to introduce a **dual root space** spanned by the **coroots**

$$\alpha_i^\vee = \frac{2}{(\alpha_i, \alpha_i)} \alpha_i. \tag{8}$$

We can then write any dual vector as

$$\lambda = \sum_i \bar{\lambda}_i \alpha_i^\vee.$$

Note that for the simply-laced groups, (8) reduces to

$$\alpha_i^\vee = \alpha_i, \tag{9}$$

so that the root space is self-dual. We then introduce the **Dynkin basis**, in which the components of λ are

$$a_i = \sum_j \bar{\lambda}_j A_{ji}, \tag{10}$$

where again A_{ij} is the Cartan matrix (7).

The importance of this basis is due to a theorem of Dynkin which says that, in the Dynkin basis, the simple roots α_i are given by the i'th row of the Cartan matrix, and have integer components.

Before we go on to talks about irreps in more detail, we need one further theorem, again due to Dynkin. This states that the **highest weight representation** of an irrep can always be selected such that the Dynkin labels a_i are non-negative integers, and indeed each such irrep is *uniquely* identified by a set of integers $(a_1, \ldots, a_l)$. Furthermore, each possible set is the highest weight of precisely one irrep.

The easiest way to see what this theorem is talking about is to look at some examples, which we do now.

5 Irreducible representations and physics

Let's just quickly recap the main ideas thus far. We pick some (simply-laced) Lie algebra $\mathfrak{g}$ and draw its Dynkin diagram. From this, we write down the Cartan matrix as explained above. Dynkin then tells us that the rank($\mathfrak{g}$) simple roots of the Lie algebra are simply given (in a suitable basis) by the rows of the Cartan matrix.

Now, to build representations we start with some highest weight state $\lambda = (a_1, \ldots, a_l)$, where the a_i are just some bunch of non-negative integers. To construct the remaining elements in the irrep we then act on this state with the 'lowering operators' $E_{-\vec{\alpha}}$ which, as we saw before, take the state $|\lambda\rangle$ to the state $|\lambda - \alpha\rangle$, providing this is itself a root.

Right, time for an example. We'll take the algebra $\mathfrak{su}(3)$, whose Cartan matrix is

$$A_{SU(3)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Dynkin's theorem then tells us that the simple roots α_i are, in the Dynkin basis,

$$\alpha_1 = (2, -1), \qquad \alpha_2 = (-1, 2).$$

In order to construct irreps, we need to pick some highest weight state, which will be some 2-vector (a_1, a_2), where the a_i are non-negative. Each choice of highest weight state will give us a different irrep.

With a little bit of foresight, let's look at what we get by choosing $a_1 = 1, a_2 = 0$. The procedure is as follows: whenever we have a root with a positive number b_j as j'th component, we can subtract b_j factors of the simple root α_j. We simply repeat this procedure until we can't go any further, i.e. all the components of the root are non-positive.

For the example above, we start with $(1, 0)$. From the discussion above we see that we can subtract α_1 from this, which gives the root $(-1, 1)$. We should now subtract α_2 from this, giving the root $(0, -1)$. Since there are no more positive components this is where we stop. We see then that the highest weight state $(1, 0)$ generates a 3-dimensional irrep of the $\mathfrak{su}(3)$ algebra, which the physics literature calls the "fundamental representation".

If we do the same thing starting with the highest weight state $(0, 1)$ then we get another 3-dimensional irrep, $\{(0, 1), (1, -1), (-1, 0)\}$ known as the "anti-fundamental representation".

In general, for the algebra $\mathfrak{su}(n + 1)$, starting with the highest weight state $(0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in the r'th position gives the $\binom{n}{r}$ representation. We can get the adjoint representation by starting with the state $(1, 0, \dots, 0, 1)$. This indicates what we already know, that for $\mathfrak{su}(n + 1)$, the tensor product of the fundamental and anti-fundamental representations gives the adjoint representation (plus a singlet). It is left as an exercise for the reader to show that, for $n = 2$ we have

$$(1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 0).$$

6 Subgroups and decompositions

An important idea in GUT model building, be it from strings or wherever, is that we should somehow be able to get Standard Model-esque gauge groups as subgroups of whatever we have in our UV theory. The mechanism for this breaking is, of course, highly model dependent, but the actual group theoretic procedure for finding irreps of the reduced Lie algebra is the same in each case, and can be expressed nicely using the language of Dynkin diagrams.

The main technique is the following: suppose I have a Lie group G with $\text{rank}(\mathfrak{g}) = l$, and some other group H with $\text{rank}(\mathfrak{h}) \leq l$. Then $H \subset G$ if the associated Dynkin diagram to the Lie algebra $\mathfrak{h}$ is a sub-diagram of that for $\mathfrak{g}$, in the sense that we can obtain the Dynkin diagram of $\mathfrak{h}$ by deleting nodes from the Dynkin diagram of $\mathfrak{g}$.

For example, we could take the Dynkin diagram corresponding to $SO(10)$ and delete node 1 (see Table 5 of [1] for nomenclature) to get the subgroup $SO(8) \times SO(2)$. Note here that $SO(2)$, as with $U(1)$, is rank 1 but has no

Dynkin diagram. In fact, for most purposes, it is taken not to have a simple Lie algebra. We could likewise delete the third node to get the subgroup $SU(3) \times SU(2) \times SU(2) \times U(1)$.

Dynkin diagrams also provide a useful way of finding the maximal subgroup of any given Lie group G. In particular, first write down the **affine** or **extended** Dynkin diagram of $\mathfrak{g}$, given in Table 16 of [1], and then delete one of the nodes from the resulting diagram. The maximal of all resulting diagrams is the maximal subgroup of G.

6.1 Useful examples for GUT models

We'll finish by talking about some useful examples which one encounters when trying to discover whether one's favourite GUT model ends up giving useful low-energy phenomenology.

The canonical example is 'breaking' $SU(5)$ to the SM gauge group $SU(3) \times SU(2) \times U(1)$, which is what we will do now.

First, take the Dynkin diagram of $\mathfrak{su}(5)$ and delete the second node, i.e. replace it with a $\mathfrak{u}(1)$. This shows us that $SU(3) \times SU(2) \times U(1)$ is a subgroup of $SU(5)$.

The Cartan matrix of $SU(5)$ is

$$A_{SU(5)} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

so that the simple roots in the Dynkin basis are given by

$$\begin{aligned} \alpha_1 &= (2,-1,0,0), \qquad \alpha_2 = (-1,2,-1,0), \\ \alpha_3 &= (0,-1,2,-1), \qquad \alpha_4 = (0,0,-1,2). \end{aligned}$$

Let's look at the irrep generated by the highest weight state $w_1 = (0,1,0,0)$. We can first subtract α_2 to get $w_2 = (1,-1,1,0)$. We can now subtract either α_1 or α_3 from this: the first choice gives $w_3 = (-1,0,1,0)$; the second choice gives $w_4 = (1,0,-1,1)$. Subtracting α_3 from w_3 or α_1 from the w_4 gives $w_5 = (-1,1,-1,1)$; subtracting α_3 from w_4 gives $w_6 = (1,0,0,-1)$. Again, we have a multiplicity of choices. Subtracting α_2 from w_5 gives $w_7 = (0,-1,0,1)$; subtracting α_4 from the w_5 or α_1 from w_6 gives $w_8 = (-1,1,0,-1)$. We can now subtract either α_4 from w_7 or α_2 from w_8 to get $w_9 = (0,-1,1,-1)$. Finally, subtracting α_3 from this gives $w_{10} = (0,0,-1,0)$, which is where we stop.

Putting this all together, we have found a 10-dimensional irrep of $SU(5)$.

Now, how does this irrep decompose under the $SU(3) \times SU(2) \times U(1)$ subgroup?

Recall that, in order to find this subgroup, we deleted the second node in the Dynkin diagram and replaced it by a $\mathfrak{u}(1)$. In terms of the roots, we have split (a_1, a_2, a_3, a_4) into $((a_3, a_4), (a_1), a_2)$, where the elements tell us whether we're talking about $SU(3)$, $SU(2)$ or $U(1)$ representations.

Under this decomposition the roots $w_{1,\ldots,10}$ which make up the 10-dimensional irrep above decompose as

$$\begin{aligned}
w_1 &= ((0,0),(0),1), \quad w_2 = ((1,0),(1),-1), \quad w_3 = ((1,0),(-1),0), \\
w_4 &= ((-1,1),(1),0), \quad w_5 = ((-1,1),(-1),1), \quad w_6 = ((0,-1),(1),0), \\
w_7 &= ((0,1),(0),-1), \quad w_8 = ((0,-1),(-1),1), \quad w_9 = ((1,-1),(0),-1), \\
w_{10} &= ((-1,0),(0),0).
\end{aligned}$$

From this, we can see that $\{w_1\}$ corresponds to the charged singlet $(1,1)_1$ of the SM; $\{w_7, w_9, w_{10}\}$ corresponds to the $SU(2)$ singlet and $SU(3)$ triplet $(\bar{3},1)_{-\frac{2}{3}}$; and the remainder $\{w_2, w_3, w_4, w_5, w_6, w_8\}$ corresponds to the state $(3,2)_{\frac{1}{6}}$.

Hence, we have shown that the 10-dimensional representation of $SU(5)$ decomposes to give us the right-handed electron, quark singlet, and quark doublet of the SM, all with the correct hypercharges!

References

[1] R. Slansky. Group theory for unified model building. *Physics Reports*, 79(1):1 – 128, 1981.

[2] R. Gilmore. *Lie groups, Lie algebras, and some of their applications.* Wiley, 1974.