The Higgs mechanism: a geometrical perspective

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March 4, 2013

Abstract

We try to understand the usual Higgs-like mechanism in a semi-rigorous mathematical style by considering vector bundles and other nice geometrical structures.

1 Introduction

The Higgs mechanism has, for obvious reasons, been in the popular media a lot recently. But being hip physicists, we knew about it before it was cool.

The usual way of learning about the mechanism talks a lot about Mexican hats, about vevs, and about things ‘eating’ degrees of freedom off of other things. In other words, we learn a lot about how the mechanism works in very specific cases (useful for talking about Standard Model phenomenology) and very little about the underlying mathematical structures. These notes aim to rectify that, and to examine how we can give more of a geometrical bent to what’s going on, thereby helping us to generalise things to situations sans Mexican hats.

I shan’t review the usual SM approach to the mechanism here: just look in your favourite QFT book.

2 Prerequisites

So if we’re going to be speaking a geometric language, we should know what the words look like. Apart from the usual ideas of manifolds, etc. we’ll be relying heavily on principal bundles and their associated vector bundles. I’ve previously given a short course on such topics, and the notes can be found on the Journal Club website [1]. We’ll also have to understand connections on bundles and something or other about their holonomy. Lie algebras, coset spaces, etc. will also appear.

A word to the wise: these notes are not intended to be an introduction to the subjects just mentioned and should not be treated as such. They exist more as a conduit to clarify some ideas which are presented to some degree in other sources, but which the author has been unable to find explicitly and fully.
3 An action and a vacuum

Let’s get things going. We want to consider a theory of some complex scalar fields $\phi$ transforming in an $m$-dimensional representation $\rho$ of some Lie group $G$, which has dimensionality $\dim(G) = n$. We denote the generators of the Lie algebra $\mathfrak{g}$ of $G$ by $T^a$, where $a = 1, \ldots, n$. In the usual approach to the Higgs mechanism, we take the Higgs field $\phi$ to transform in the fundamental representation of $G$. For example, in electroweak theory one takes $G = U(2)$ and $\phi$ to be a complex doublet. However, this setup doesn’t allow for instanton solutions, which require $\phi$ transforming in the adjoint representation of $G$, and is the case that comes up in, e.g. Seiberg-Witten. This is also the case we’ll consider when we look at D-branes in type IIB later on.

Geometrically, what we’re looking at here is a principal $G$-bundle $P(M, G)$ over some spacetime manifold $M$ (in QFT applications it’s just taken to be Minkowski space, but we like to keep things general.) The connection 1-form $A = A^a T^a$ is valued in the Lie algebra $\mathfrak{g}$. We then have some associated vector bundle of complex dimension $m$. Recall that this is just saying that, for $u \in P(M, G)$, a $G$-transformation on the principal bundle $u \mapsto ug$ induces an action $\phi \mapsto \rho(g)\phi$ on the vector bundle, where $\phi \in \mathbb{C}^m$.

Let’s write down an action for this theory. We’re going to have our usual kinetic term for the gauge fields, a gauge-covariant kinetic term for the scalars, and some scalar potential $V(\phi)$ which is invariant under the action of $G$, i.e.

$$V(\rho(g)\phi) = V(\phi) \quad \forall g \in G. \quad (1)$$

The full Lagrangian then takes the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |D\phi|^2 - V(\phi). \quad (2)$$

Since it may come in handy in the future, we define the $G$-covariant derivative as $D = \partial + igA$ for some coupling $g$ so that, with everything¹ written out explicitly,

$$D\phi = d\phi + igA^a \rho(T^a)\phi. \quad (3)$$

Note that, as we require, the whole Lagrangian (2) is $G$-invariant.

We can define a Higgs vacuum of the theory (2) to be the set of solutions to the equations

$$V(\phi) = 0, \quad D\phi = 0. \quad (4)$$

In Section 1.4 of [2], where the Georgi-Glashow model is considered, this is shown to be a consequence of the condition that the solutions give vanishing energy density.

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¹Except for the representation indices: we won’t write those anywhere to avoid clutter, but it should be obvious where they go.
4 The vacuum manifold

Let’s concentrate first on the first of the conditions (4), which tells us that a vacuum solution $\phi_0$ of the theory (2) should satisfy $V(\phi_0) = 0$. The set of all such solutions defines the so-called vacuum manifold

$$M_0 := \{ \phi : V(\phi) = 0 \}. \quad (5)$$

Now, from the definition of the potential $V(\phi)$, we can see that the vacuum manifold (5) is invariant under the action of the gauge group $G$. If the potential is ‘sufficiently well-behaved’, for a certain value thereof$^2$, then it’s reasonable to assume that in fact the group $G$ acts transitively on $M_0$. That is, I can get from any vacuum solution $\phi_0 \in M_0$ to any other vacuum solution $\phi' \in M_0$ by the action of some group element $g \in G$.

Now, it may be the case that, given some vacuum solution $\phi_0$, not every element $g \in G$ moves $\phi_0$ to some other solution. In particular, we can look at the stabilizer (or Little group) of $\phi_0$ in $G$, which we’ll call $H$. That is

$$H = \text{Stab}_G(\phi_0) = \{ g \in G : \rho(g)\phi_0 = \phi_0 \}. \quad (6)$$

Since $G$ acts transitively on $M_0$, we can show that the stabilizer of any point is isomorphic to the stabilizer of any other point in $M_0$, so $H$ is really the isotropy subgroup of $G$. Thus, we can write

$$M_0 = G/H, \quad (7)$$

as a set of right cosets. Hence, the moduli space of Higgs vacua is some coset space of the gauge group $G$.

5 The unbroken gauge group

Now that we’ve been able to describe the vacuum manifold, which we defined via the first condition in (4), let’s move on to the consider the second of those conditions defining the Higgs vacuum, namely (in components)

$$D_\mu \phi = 0, \quad (8)$$

which should be satisfied for any vacuum solution $\phi \in M_0$. Let’s now look at the combination

$$[D_\mu, D_\nu] \phi.$$

From (8) it’s clear that this should vanish, since the covariant derivatives annihilate the vacuum solution. On the other hand, we know from the general theory of connections on principal bundles that this combination defines the curvature $F$ of the connection

$^2$We really just require that there are no accidental global symmetries which would expand the symmetry group of $M_0$.\hfill
\[ [D_\mu, D_\nu] \phi = F_{\mu\nu} \phi, \]  

so we deduce that, for any given vacuum solution \( \phi_0 \in M_0 \),

\[ F_{\mu\nu}^a \rho(T^a) \phi_0 = 0. \]  

But this is really telling us something special. Given some vacuum solution \( \phi_0 \), let’s split the generators \( T^a \) into two sets: those which satisfy \( \rho(T^a) \phi_0 = 0 \), which are referred to in the physics literature as the unbroken generators; and those which satisfy \( \rho(T^a) \phi_0 \neq 0 \), which are referred to as the broken generators. Then equation (10) is saying that the curvature 2-form \( F \) only takes values in the subalgebra generated by the ‘unbroken’ generators.

Let’s look a little closer at these ‘unbroken’ generators, which we’ll label now \( T^i \). We can exponentiate and consider some group element \( h \in G \),

\[ h = \exp(\alpha^i T^i). \]

This isn’t a completely generic element of \( G \) however, because (10) tells us that

\[ \rho(h) \phi_0 = \phi_0, \]

or, in other words, \( h \) lives in the subgroup of \( G \) which stabilizes the vacuum solution \( \phi_0 \). But we’ve seen this subgroup before: it’s precisely the isotropy subgroup \( H \) we came across when looking at the vacuum manifold!

6 The Ambrose-Singer theorem and reduction of the structure group

We should take stock for a moment and see where we are. The main ingredients we started with were some Lie group \( G \) and some scalar fields \( \phi \) transforming in a particular representation of it. We then threw a \( G \)-invariant scalar potential term into the mix, the set of zeroes of which we saw had the structure of a coset manifold \( M_0 = G/H \). We then saw that, by picking out a particular vacuum configuration, i.e. a particular solution of (4), the curvature 2-form was forced to take values in the Lie algebra \( \mathfrak{h} \subset \mathfrak{g} \) of the group \( H \).

Taking this as our starting point, our aim is to show that, by selecting a particular vacuum configuration, we’ve reduced our original principal \( G \)-bundle \( P(M, G) \) to a principal \( H \)-bundle \( P'(M, H) \).

Our first step is something called the Ambrose-Singer theorem, which relates the holonomy of a connection (how vectors change when they’re transported on loops) to the curvature. We won’t go into the full technical details of the theorem (see, e.g. [3]) but paraphrased it tells us that if the curvature takes values in some Lie algebra \( \mathfrak{h} \) of \( H \), then the holonomy of the connection \( A \) is contained in a subgroup \( H \subset G \).
Now, the next step from here would be to say something like “If the holonomy of the connection is contained in some subgroup $H \subset G$, then there exists a map $f : P(M, G) \to P(M, H)$ which reduces the structure group.” Unfortunately, though I believe this statement to be true on various unrigorous grounds, I’ve yet to find it written down explicitly.

What this would mean, at the end of the day, is that by selecting a particular vacuum state from $\mathcal{M}_0 = G/H$, we have ‘broken’ the gauge group under which the fields transform from $G$ to $H$.

7 Higgs and D-branes

7.1 Introductory discussion

For those of us that begin to feel a little light-headed at the thought of a QFT textbook, we can take a more “natural” example of this process of ‘Higgsing’, which can be found in the study of D-branes in string theory. The following description is a little hand-wavy, but we’ll make it more precise below.

Let’s take a stack of $N$ D3-branes in IIB, which has oriented strings. By the usual reasoning, we can associate group-theoretic labels, the so-called Chan-Paton factors, with the ends of these strings which transform in the fundamental or anti-fundamental (i.e. an $N$-dimensional representation) of $U(N)$. What this means is that we can associate with our stack a $U(N)$ gauge bundle over the four-dimensional worldvolume of the branes. This is the familiar statement that D-branes have gauge theories living on them. It’s really saying that in the IR (so we only keep the massless modes of the theory), a theory of strings moving on some stack of D3-branes looks like a $U(N)$ gauge theory in four dimensions.

Now, consider what happens when we split the stack. We take some number $K$ of the D3-branes and move it some finite distance away. Now the strings that stretch between the two stacks get some tension and become massive, so that they drop out of the massless spectrum and don’t survive to the IR. What’s left is the strings which live solely on the stack of $K$ D3-branes or the stack of $(N-K)$ D3-branes. In the IR this looks like a $U(K) \times U(N-K)$ gauge theory in four dimensions.

So what we’ve done here is reduce our gauge bundle

$$E(M, U(N)) \to E(M, U(K) \times U(N-K)).$$

In our earlier language, $G = U(N)$ and $H = U(K) \times U(N-K)$, so we should really expect there to be some vacuum manifold associated with this Higgsing, which in this case is

$$\mathcal{M}_0 = \frac{U(N)}{U(K) \times U(N-K)}.$$

The manifold $\mathcal{M}_0$ has various nice properties. It’s actually the space of $K$-linear subspaces (i.e. lines if $K = 1$ and planes if $K = 2$) through the origin in $\mathbb{C}^N$, also known as the complex Grassmannian $\text{Gr}(K, \mathbb{C}^N)$. All we really
need to know though is that it has dimension \(2K(N - K)\), is compact, and is Kähler.

### 7.2 Something a bit more precise

The above discussion, despite giving the general idea of what’s going on, is a little bit too wordy to be of much use, so let’s try to make things more precise. We mostly follow the treatment by Witten [4].

As mentioned above, we’re interested in type IIB string theory in the presence of a stack of \(N\) D3-branes, labelled by \(a = 1, \ldots, N\). We’re going to be interested in the open string sector of this theory, described by oriented strings which begin and end on the stack, which we take to be at a position \(x^j = 0\) in the ten-dimensional spacetime, where \(j = 4, \ldots, 9\) denote the spacetime coordinates transverse to the stack. We use \(x^s\) with \(s = 0, \ldots, 3\) to denote the worldvolume coordinates of the stack.

We can think of the open strings as describing the excitations of the D3-branes. If we’re only interested in the low-energy excitations of the stack, then these are described simply by quantising the open strings and looking at the massless spectrum. Since each open string can begin or end on any one of \(N\) D3-branes, we need to include some Chan-Paton factors \(\lambda_{ab}\) when describing each state\(^3\).

If we look back at the spectrum of the open superstring [5] we see that the massless states consist of a ten-dimensional \(U(N)\) vector supermultiplet. The effective action for this theory is then given simply by ten-dimensional SYM with a \(U(N)\) gauge group

\[
\mathcal{L}_{\text{SYM}} = -\frac{1}{4} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{i}{2} \text{Tr}(\bar{\psi} \Gamma^\mu D_\mu \psi).
\] (12)

Here \(\mu, \nu = 0, \ldots, 9\) are spacetime coordinates, the covariant derivative is given by \(D\psi = d\psi - i[A, \psi]\), and the field strength \(F = dA - i[A, A]\). One can argue [4] that the entire massless spectrum is independent of the transverse coordinates, so that we can describe this theory equivalently by a theory obtained by dimensional reduction of (12) on \(T^6\).

Let’s concentrate solely on the bosonic sector for a moment. Recall that the gauge fields are really some connection on a \(U(N)\) principal bundle, so we implicitly have some generators \(T^a\) in the adjoint representation, i.e. \(A = A^a T^a\) for \(a = 1, \ldots, N\). Hence, after dimensional reduction, we will get some four-dimensional gauge fields \(A_r\) and six scalars \(X^i \sim A_i\) which transform in the adjoint representation of \(U(N)\).

Reduction of the bosonic part of (12) will then give us the four-dimensional action

\(^3\)The barred index corresponds to the start of the string, the unbarred index to the end, where such words as “start” and “end” are defined with respect to the orientation of the string.
\[ \mathcal{L} = -\frac{1}{4} \text{Tr} \left( F_{rs} F^{rs} \right) + \frac{1}{2} \sum_{i=4}^{9} \text{Tr} (DX^i)^2 - \frac{1}{4} \sum_{i,j=4}^{9} \text{Tr} \left( [X^i, X^j]^2 \right). \quad (13) \]

Including the reduction of the fermionic terms in (12) will give us the famous four-dimensional \( \mathcal{N} = 4 \) SYM theory\(^4\).

Let’s compare this action (13) with the general form of an action with Higgs fields (2) with which we started. We see that it takes exactly the same form, with potential term

\[ V = \frac{1}{4} \sum_{i,j=4}^{9} \text{Tr} \left( [X^i, X^j]^2 \right), \quad (14) \]

which is invariant under the action of \( U(N) \). We can define a vacuum manifold by the condition

\[ [X^i, X^j] = 0. \quad (15) \]

Writing the \( X^i \) as \( N \times N \) matrices, the condition (15) tells us that they should be simultaneously diagonalisable. Hence, we can write any state in our vacuum manifold as

\[ X^i = \text{diag} \left( x^i_{(1)}, \ldots, x^i_{(N)} \right), \quad (16) \]

for some constants \( x^i_{(a)} \) which represent the positions of the D3-branes in the six-dimensional transverse space.

This is basically telling us that the vacuum configurations of the D3-brane system are given by stacks of the \( N \) D3-branes placed at different positions in the transverse space.

In particular, let’s consider a situation with \( (N - 1) \) of the D3-branes at some position \( x^i_{0} \) and the remaining one at some \( x^i_{N} \). This corresponds to a point in the vacuum manifold

\[ X^i = \text{diag} \left( x^i_{0}, \ldots, x^i_{0}, x^i_{N} \right). \quad (17) \]

The isotropy group of this point is just going to be \( U(N - 1) \times U(1) \subset U(N) \), where the action is the obvious one. This is the \( K = 1 \) case of the situation we talked about earlier, but hopefully we now have a better idea of what’s going on.

To finish, we can describe the full moduli space of vacua for the four-dimensional \( \mathcal{N} = 4 \) theory (13). Each point in moduli space (i.e. each vacuum configuration) corresponds in the stringy picture to stacks of D3-branes at positions described by (16). Overall there are \( 6N \) possible real numbers \( x^i_{\alpha} \), so that any point in moduli space can be reached by any other via the action of

\(^4\text{\( \mathcal{N} = 4 \) SYM in four dimensions also contains an } F \wedge F \text{ term which reduces from a topological term in the ten-dimensional action, but we shan’t include that here.}\)
some translation in $\left(\mathbb{R}^6\right)^N$. However, since the D3-branes are indistinguishable from each other, we can relabel them and get the same vacuum configuration. Hence, we see that the overall moduli space for this theory is

$$\mathcal{M}_{\text{SYM}} = \frac{\left(\mathbb{R}^6\right)^N}{S_N},$$

where $S_N$ here denotes the symmetric group on $N$ objects.

At a generic\(^5\) point in this moduli space, the gauge symmetry will be broken from $U(N)$ to $U(1)^N$. However, if the point lies on some surface of codimension one or higher, we will get enhancements of this gauge group to, e.g. $U(1)^{N-K} \times U(K)$.

References


\(^5\)Where generic takes on its technical meaning, namely: the set of points where the condition is not satisfied has codimension one.