An Introduction to Calabi-Yau Manifolds

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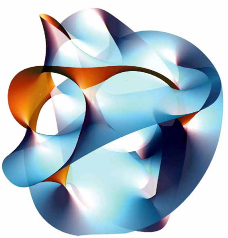

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- Cardoso et al. *Non-Kähler String Backgrounds and their Five Torsion Classes* [arXiv: hep-th/0211118]
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Motivation

- Use in superstring theory.
- Superstrings conjectured to exist in 10d: $M_4 \times \mathrm{CY}_3$. i.e. 4d Minkowski that we are aware of plus an additional 6 compact dimensions (CY_3 is 3 cx dims or 6 real dims).
- Compactification of extra dimensions on CY mfolds is popular as it leaves **some** of the original SUSY unbroken (specifically, 2^{1-n} of the original SUSY is unbroken if we compactify on a CY mfold with holonomy group $SU(n)$).[1]
- Several other motivations for studying these: F-theory compactifications on CY 4-folds allow you to find many classical solutions in the string theory landscape[2].
- First attempts at obtaining standard model from string theory used the now "standard" compactification of $E_8 \times E_8$ heterotic string theory. In such compactifications, $\#$ generations $= \frac{1}{2}|\chi|$ where χ is Euler characteristic. $\therefore$ for 3 generation model, want $\chi = \pm 6$. Return to this later[3].

[1] Want some SUSY to get broken at low energy since we don't observe it but we do want $\mathcal{N} = 1$ SUSY to remain unbroken at low energies as we need it to solve problems e.g. mass of scalars etc (this is the reason we consider SUSY to begin with). However, we want all others to be broken at low energies since they're not realistic e.g. $\mathcal{N} = 2$ is not chiral. At high energies, these extra SUSYs (if they exist) could be unbroken and present.

[2] See Will for details.

[3] Return to in Hodge Diamond discussion.

Aims of Talk

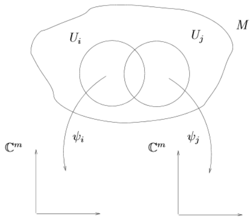
- Complex Manifolds
- Kähler Manifolds
- Homology and Cohomology
- Chern Classes
- CY Mfolds

Complex Mfolds (1)

We review the two possible constructions of complex mfolds and then give the decomposition of the complexified tangent and cotangent bundles, that in turn, allows us to define (p, q) forms on M that will be useful later on.

Complex Mfolds (2): Construction (a)

- M is differentiable mfold covered by open sets $\{U_a\}_{a \in A}$
- Each U_a has a corresponding coordinate map $z_a : U_a \to \mathbb{C}^n$ to an open subset of $\mathbb{C}^n$
- On non-trivial intersections $U_a \cap U_b \neq \emptyset$, the transition functions $z_a \cdot {z_b}^{-1} : z_b(U_a \cap U_b) \to z_a(U_a \cap U_b)$ are holomorphic (c.f. smooth for a smooth real mfold).
- Essentially, holomorphic transition fns $\Rightarrow$ cx mfold.



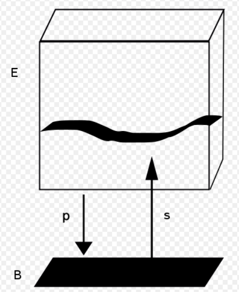
Complex Mfolds (3): Construction (a)

- So a cx mfold will look, *at least locally*, like $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.
- Construction (a) makes it clear that any cx mfold is also a real mfold (can be made explicit by expanding the n cx coordinates in terms of real and imaginary parts [$2n$ coords]).
- However, converse not all true i.e. not all real mfolds are cx mfolds.
- Becomes apparent when consider Construction (b).

Complex Mfolds (4): Construction (b)

Preliminaries:

- Let M be a real n-fold with tangent and cotangent bundles TM and T^*M resp.
- Recall that a fiber bundle consists of the data (E, B, p, F) where the bundle projection $p : E \to B$. We require that for every open $U \subset B$, $p^{-1}(U) \subset E$ is homeomorphic to $U \times F$ such that the natural projection on the first factor returns U i.e. we get the standard commuting triangle.
- A section of a bundle is a continuous map $s : B \to E$ defined s.t. $\forall x \in B$, $p(s(x)) = x$.
- e.g. if E is a vector bundle, a section of E is an element of the vector space E_x lying above each $x \in B$ i.e. it picks out a particular vector.



Complex Mfolds (5): Construction (b)

- Sections of TM (resp T^*M) are tangent vector fields and covector fields resp.
- Sections of the tensor product bundle $\otimes^k TM \otimes^l T^*M$ are tensor fields of type (k, l).
- The space of type (k, l) tensor fields is denoted $\Gamma(\otimes^k TM \otimes^l T^*M)$.

Complex Mfolds (6): Construction (b)

- Let M be real $2n$-fold. Define the *almost complex structure* $J \in \Gamma(TM \otimes T^*M)$ which satisfies $J^a_b J^b_c = -\delta^a_c$.
- For any $v \in \Gamma(TM)$, $J^2 v = -v \Rightarrow J$ is a generalisation of multiplication by $\pm i$
- J gives the structure of a complex vec. space to each $T_p(M) \forall p \in M$
- (M, J) is called an *almost complex* $2n$*-fold*.

Complex Mfolds (7): Construction (b)

- Define the *Nijenhuis tensor* $N \in \Gamma(TM \otimes^2 T^*M)$ by its action on $v, w \in \Gamma(TM)$ by
  $N(v, w) = -J^2[v, w] + J[Jv, w] + J[v, Jw] - [Jv, Jw]$
- If (M, J) is an almost cx $2n$-fold with $N = 0$ then J is called a *cx structure* and M a *cx n-fold*.
- Two constructions are equivalent since holomorphic transition functions $\Leftrightarrow$ J integrable $\Leftrightarrow$ $N = 0$ $\Leftrightarrow$ J complex structure.
- It is this condition of integrability of J being satisfied, that allows M to be covered by cx coordinates.

Complex Mfolds (8): Bundle Decomposition

- Take a cx n-fold (M, J) (J cx structure).
- $J_p : T_p(M) \to T_p(M)$ is an endomorphism of tangent spaces.
- If we complexify the tangent spaces $T_p(M) \mapsto T_p(M) \otimes \mathbb{C}$ then J_p extends naturally to $J_p : T_p(M) \otimes \mathbb{C} \to T_p(M) \otimes \mathbb{C}$.
- Since $J^2 = -\mathbb{I}$, the evalues of J in $T_p(M) \otimes \mathbb{C}$ are $\pm i$
- $\Rightarrow \exists$ 2 espaces of J with evalues $\pm i$ (denoted $T_p^{1,0}(M)$, $T_p^{0,1}(M)$ resp).
- $\Rightarrow T_p(M) \otimes \mathbb{C} = T_p^{1,0}(M) \oplus T_p^{0,1}(M)$ where $T_p^{1,0}(M)$, $T_p^{0,1}(M)$ are conjugate to each other and isomorphic to $\mathbb{C}^n$
- p arbitrary so applies to **complexified** tangent bundle i.e. $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. $T^{1,0}M$, $T^{0,1}M$ called holomorphic and anti-holomorhpic tangent bundles resp.
- N.B. Sections of complexified bundles are complex-valued
- This decomposition allows us to project out holo & anti-holo pieces e.g. $S^a = S^\alpha + S^{\bar{\alpha}}$ (Latin indices are real, Greek indices are complex).

Complex Mfolds (9): (p, q)-forms

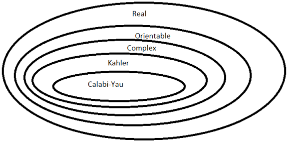
- k^{th} wedge power of T^*M denoted $\Lambda^k T^*M$. Smooth sections are called k-forms. Space of **cx-valued** k-forms denoted $A^k(M)$.
- Given the decomposition of the complexified cotangent bundle $T^*_{\mathbb{C}}M = T^{*1,0}_{\mathbb{C}}M \oplus T^{*1,0}_{\mathbb{C}}M$, we define (p, q)-forms as forms with p holo and q anti-holo indices.
- i.e. a (p, q)-form is a smooth section of $A^{p,q} := \Gamma(\wedge^p T^{*1,0}M \wedge^q T^{*0,1}M)$
- e.g. $T_{\alpha_1 \dots \alpha_p \bar{\alpha}_{p+1} \dots \bar{\alpha}_{p+q}} dz^{\alpha_1} \dots dz^{\alpha_p} d\bar{z}^{\bar{\alpha}_{p+1}} \dots d\bar{z}^{\bar{\alpha}_{p+q}} \in A^{p,q}$
- $A^k = \displaystyle\bigoplus_{p+q=k} A^{p,q}$ (take a cx-valued k-form written in real coords and expand in terms of holo and anti-holo coords ($dx^a = dz^a + d\bar{z}^{\bar{a}}$) and take all combinations of $dz, d\bar{z}$'s).

Complex Mfolds (10): (p, q)-forms

- (p, q)-forms will be useful later on.
- Exterior derivative also decomposes as $d = \partial + \bar{\partial}$
- $\partial : A^{p,q} \to A^{p+1,q}, \quad \bar{\partial} : A^{p,q} \to A^{p,q+1}$
- $d^2 = 0 \Rightarrow \partial^2 = 0, \bar{\partial}^2 = 0, \partial\bar{\partial} + \bar{\partial}\partial = 0.$

Kähler Mfolds (1): A Hermitian Metric

- Kähler mfolds are a subclass of cx mfolds and, as such, are naturally oriented.
- In addition to J, Kähler mfolds have a Hermitian metric g ($+$ associated connection) and can thus be denoted by the triplet (M, g, J).
- Hermitian metric g satisfies $g(v, w) = g(Jv, Jw) \Rightarrow g_{ab} = J^c_a J^d_b g_{cd}$ J is block-diagonal with holo e-values i and anti-holo e-values $-i$ i.e. $g_{ab} = g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\beta}$
- It's a symmetric, $+$ve definite inner product $T^{1,0}M \otimes T^{0,1}M \to \mathbb{C}$
- Symmetry means that line element is $ds^2 = 2g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}}$



Kähler Mfolds (2): Fundamental 2-forms

- Given a Hermitian g, we can define a fundamental 2-form ω by $\omega(v, w) = g(Jv, w) \forall v, w \in \Gamma(TM)$
- In real cpts, $\omega_{ab} = \frac{1}{2} J_a^c g_{cb}$.
- In cx cpts, $J = \mathrm{diag}(i, \ldots, i, -i, \ldots, -i) \Rightarrow \omega_{ab} = \frac{1}{2} i g_{\alpha\bar{\beta}} - \frac{1}{2} i g_{\bar{\alpha}\beta}$
- As a differential form, $\omega = i g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}}$ i.e. actually a $(1, 1)$-form.
- So far, this defines a **Hermitian mfold**.

Kähler Mfolds (3): Kähler $\subset$ Hermitian

- In order for the Hermitian mfold (M, g, J) to be a Kähler mfold, we require ω to be closed; $d\omega = 0$.
- Then ω is a Kähler form, g a Kähler metric and (M, g, J) a Kähler mfold.
- $d\omega = 0 \Leftrightarrow \partial_\gamma g_{\alpha\bar{\beta}} = \partial_\alpha g_{\gamma\bar{\beta}} \Leftrightarrow g_{\alpha\bar{\beta}} = \partial_\alpha \bar{\partial}_{\bar{\beta}} K(z, \bar{z})$
- $K(z, \bar{z})$ is called the *Kähler potential*. K isn't unique: Kähler transformations $K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + \bar{f}(\bar{z})$ give same metric.
- Aside: In fact, $\omega^n \propto$ volume form on M confirming natural orientability.

Kähler Mfolds (4): 2 out of 3 Theorem

- Since, in cx coords, $J^a_b = \begin{pmatrix} i\delta^\alpha_\beta & 0 \\ 0 & -i\delta^{\bar{\alpha}}_{\bar{\beta}} \end{pmatrix}$, we can show that all cx mfolds satisfy $\nabla_c J^a_b = \partial_c J^a_b + i\Gamma^a_{cd} J^d_b - i\Gamma^d_{cb} J^a_d = 0$
- The 2 out of 3 theorem states that given any two of $\nabla g = 0, d\omega = 0, \nabla J = 0$, the third is always true as well.
- Note that $d\omega = 0 \Rightarrow (\partial + \bar{\partial})\omega = 0 \Rightarrow \partial\omega = \bar{\partial}\omega = 0$ They're independently zero since $\partial, \bar{\partial}$ map $A^{p,q}$ to different spaces (no possibility of cancellation).
- Consequently, expanding in cpts we find $\partial_\alpha g_{\beta\bar{\gamma}} = \partial_\beta g_{\alpha\bar{\gamma}}, \bar{\partial}_{\bar{\alpha}} g_{\beta\bar{\kappa}} = \bar{\partial}_{\bar{\kappa}} g_{\beta\bar{\alpha}}$ ($g = \partial\bar{\partial}K$ and partial derivatives commute).
- Only unmixed cpts of connection (Christoffel symbols) are non-zero. Important as it means no mixing of holomorphic and anti-holomorphic pieces of tensor fields under parallel transport.

(Co-)homology (1): Homology Basics

- Standard construction of homology takes some object X e.g. topological space, manifold, etc. on which we define a sequence of abelian groups $A_0, A_1, \ldots$ connected by homomorphisms $\partial_n : A_n \to A_{n-1}$ which are nilpotent ($\partial_{n-1} \cdot \partial_n = 0$).
- This forms what's known as the *chain complex* $\ldots \stackrel{\partial_{n+1}}{\to} A_n \stackrel{\partial_n}{\to} A_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots \stackrel{\partial_2}{\to} A_1 \stackrel{\partial_1}{\to} A_0 \stackrel{\partial_0}{\to} 0$
- The nilpotency of the so-called 'boundary operators" is such that $\mathrm{Im}(\partial_{n-1}) \subset \mathrm{Ker}(\partial_n)$
- This means we can define the n^{th} homology group as the quotient $H_n(X) := \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n-1})}$
- Often we use the notation $Z_n(X) := \mathrm{Ker}(\partial_n)$ (n^{th} cycle groups) and $B_n(X) := \mathrm{Im}(\partial_n)$ (n^{th} boundary groups) in which case $H_n(X) = \frac{Z_n(X)}{B_n(X)}$

(Co-)homology (2): Cohomology Basics

- Cohomology works in a similar fashion. Difference is the homomorphisms work in direction of increasing n i.e. $\partial_n : A_n \to A_{n+1}$
- We set up what's known as a cochain complex
  $0 \stackrel{\partial_0}{\to} A_1 \stackrel{\partial_1}{\to} A_2 \stackrel{\partial_2}{\to} \dots \stackrel{\partial_{n-2}}{\to} A_{n-1} \stackrel{\partial_{n-1}}{\to} A_n \stackrel{\partial_n}{\to} \dots$
- We define the n^{th} cocycle and coboundary groups by $Z^n(X) := \mathrm{Ker}(\partial_n)$ and $B_n(X) := \mathrm{Im}(\partial_n)$ resp. We note $B_{n-1} \subset Z_n$ by nilpotency of ∂_n
- The n^{th} cohomology group is then $H^n(X) := \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n-1})} = \frac{Z^n(Z)}{B^{n-1}(X)}$
- A theorem of de Rahm tells us homology and cohomology are dual to one another.
- Makes sense since homology $\to$ cohomology involves replacing each A_n by its dual A_n^* and the homomorphisms $\partial_n : A_n \to A_{n-1}$ are replaced by their transpose $\partial_n^T : A_{n-1} \to A_n$ (appear to have relabelled this map to ∂_{n-1}^T in above cochain complex).

(Co-)homology (3): de Rahm

- Begin by reviewing situation on real n-fold M i.e. take $X = M$ in the above.
- Since the exterior derivative takes k-forms to $(k+1)$-forms i.e. $d : A^k \to A^{k+1}$, it's possible to construct a cochain complex:
- $0 \xrightarrow{d} A_1 \xrightarrow{d} A_2 \xrightarrow{d} \dots \xrightarrow{d} A_{n-1} \xrightarrow{d} A_n \xrightarrow{d} 0$
  The 0 on RHS is because can't have $(n+1)$-forms on an n-fold.
- k^{th} cocycle group Z_k is just group of closed k-forms
- $H^k_{\text{DR}} = \frac{Z^k}{B^{k-1}}$
- H^k_{DR} is a quotient group consisting of equivalences classes (cohomology classes) of closed k-forms where two closed k-forms are equivalent if they differ by an exact form i.e. $\omega_k \sim \omega_k + d\alpha_{k-1}$

(Co-)homology (4): Poincare Duality and Betti Numbers

- k^{th} Betti number defined as $b_k = \dim\left(H^k(M)\right)$
- Poincare duality states $H^k(M) \simeq H^{n-k}(M)$ and thus $b_k = b_{n-k}$ for an n-fold M.

Dolbeaut Cohomology (1)

- Previous examples (e.g de Rahm) were for real mfolds. Dolbeaut cohomology is for cx mfolds.
- Very similar but this time we use the operator $\bar{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$. This involves (p,q) forms which rely on the holomorphic structure, which in turn relies on the existence of J (need $\pm i$ e-spaces to define a holomorphic split of tangent/cotangent bundles).
- We find $H^{p,q}_{\bar{\partial}}(M) = \frac{Z^{p,q}_{\bar{\partial}}(M)}{\bar{\partial}(A^{p,q-1}(M))}$
- On a compact Kähler mfold (and hence CY mfold), the decomposition of k^{th} wedge power of complexified cotangent bundle A^k into direct sums of $A^{p,q}$ with $p+q=k$ extends to cohomology i.e. $H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(M)$

Dolbeaut Cohomology (2)

- Analogous to Betti numbers, we introduce Hodge numbers $h^{p,q} = \dim_{\mathbb{C}}(H^{p,q}_{\bar{\partial}}(M))$
- For compact, connected mfolds (e.g. CY), $h^{p,q}$ are finite and can be arranged in Hodge diamond
- Draw Hodge diamond for $\dim_{\mathbb{C}}(M) = 3$ (as we'll later be interested in CY 3-folds.
- Can introduce an operator $\bar{\partial}^* : A^{p,q}(M) \to A^{p,q-1}(M)$. Then $\bar{\partial}$-Laplacian is $\Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^*)^2$.
- A (p,q)-form ψ is harmonic if $\Delta_{\bar{\partial}}\psi = 0$

Dolbeaut Cohomology (3)

- Theorem of Hodge says every (p,q)-form φ can be decomposed as $\varphi = h + \bar{\partial}\psi + \bar{\partial}^*\eta$ where $h \in \mathcal{H}^{p,q}(M), \psi \in A^{p,q-1}(M), \eta \in A^{p,q+1}(M)$.
- If we want φ closed ($\bar{\partial}\varphi = 0$) then need $\bar{\partial}h + \bar{\partial}^2\psi + \bar{\partial}\bar{\partial}^*\eta = 0$
- But $\Delta_{\bar{\partial}} h = 0 \Rightarrow \bar{\partial}h = 0$ and $\bar{\partial}^2\psi = 0$ automatically. Thus we require $\bar{\partial}^*\eta = 0$.
- Thus $Z^{p,q}_{\bar{\partial}}(M) = \mathcal{H}^{p,q}(M) \oplus \bar{\partial}A^{p,q-1}(M) \Rightarrow \mathcal{H}^{p,q}(M) = \frac{Z^{p,q}_{\bar{\partial}}(M)}{\bar{\partial}A^{p,q-1}(M)} \simeq H^{p,q}_{\bar{\partial}}(M)$
- Also have the following identities
$$\sum_{p+q=r} h^{p,q} = b_r, \quad \sum_{p,q}(-1)^{p+q} h^{p,q} = \sum_r (-1)^r b_r = \chi(M)$$

Bundle Valued Cohomology

- To define CY mfolds, we need Chern classes, and this needs bundle valued cohomology.
- *Vector bundle of rank r* over a base space M (cx mfold) is where the fiber F is isomorphic to $\mathbb{C}^r$ i.e. $F \simeq \mathbb{C}^r$
- This means we can treat F as a mfold in own right and introduce coords $\xi = (\xi^1, \ldots, \xi^r) \in \mathbb{C}^r$ on some patch U of F.
- If we want a *holomorphic* vector bundle then we need additional structure: we require the transition fns between different coordinate systems on non-trivial patch intersections $U \cap U' \neq \emptyset$ must be $r \times r$ matrices of holomorphic fns.
- This is a cx mfold so set up a bundle valued Dolbeaut cohomology using operator $\bar{\partial} : A^{p,q}(V) \to A^{p,q+1}(V)$.
- N.B. $A^{p,q}$ is space of (p, q)-forms whilst $A^{p,q}(V)$ is space of (p, q)-forms valued in V i.e. element of $A^{p,q}(V)$ is a vector with r components, each of which is a (p, q)-form.
- Bundle valued cohomology groups are $H^{p,q}_{\bar{\partial}}(M, V) = \frac{Z^{p,q}_{\bar{\partial}}(M,V)}{\bar{\partial}(A^{p,q-1}(M,V))}$

Chern Classes (1)

- Given a Kähler metric, we can define a $(1,1)$-form Θ by $\Theta^j_i = g^{j\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^l$
- Θ is curvature 2-form for $T^{1,0}M$ (holomorphic tangent bundle).
- The Chern form/total Chern class is
$$c(M) = 1 + \sum_{i \geq 1} c_i(M) = \det(1 + \frac{it}{2\pi}\Theta)|_{t=1} = 1 + t\phi_1(g) + t^2\phi_2(g) + \dots)|_{t=1}$$
- $dc(M) = 0$ since $c(M)$ is a det (scalar). Since $d\phi_i$ are all forms of different rank, they can't cancel so must vanish separately $\Rightarrow d\phi_i(g) = 0$
- d-closed and $\bar{\partial}$-closed are equivalent on Kähler mfold[4]. Hence $[\phi_i] \in H^{i,i}_{\bar{\partial}}(M, \mathbb{C})$ (treat M as cx mfold) or $[\phi_i] \in H^{2i}_{\mathrm{DR}}(M, \mathbb{R})$ (treat M as real mfold). Or rather, $[\phi_i] \in H^{i,i}_{\bar{\partial}}(M, \mathbb{C}) \cap H^{2i}_{\mathrm{DR}}(M, \mathbb{R})$
- $[\phi_i(g)]$ is independent of g. Changing g changes Θ by an exact form but since we are quotienting by exact forms, this will change the representative but we will stay in same class.
- $\phi_i(g)$ is a representative for $c_i(M)$
- $c_i(M)$ is the i^{th} Chern class of M. Often refer to $\phi_i(g)$ as i^{th} Chern class.

[4](14.138) Blumnehagen $\Rightarrow$ d- and $\bar{\partial}$-harmonic equivalent on Kähler mfold. But $\nabla_d \alpha = 0 \Rightarrow d\alpha = 0$ and similarly $\nabla_{\bar{\partial}}\alpha = 0 \Rightarrow \bar{\partial}\alpha = 0$ (see back of p.462 Blumenhagen). Thus d- and $\bar{\partial}$- closed equivalent on Kähler.

Chern Classes (2)

- Chern classes can be generalised to any vector bundle V over M where we'd need to use Θ as the curvature 2-form of V and use the hermitian metric h. When we talk about Chern class of M we mean Chern class of $T^{1,0}(M)$ and that's why we used that particular Θ on last slide.
- Nakahara gives a straightforward prescription for evaluating arbitrary Chern classes but to save time don't give it here.
- We find $c_0(M) = [1], c_1 = \left[\frac{i}{2\pi}\Theta^i_i\right] = \left[\frac{1}{2\pi}\mathrm{Tr}\Theta\right], \ldots$
- As far as defining CY mfolds goes, we only need c_1.
- Can be shown that $c_1(M) = -\left[\frac{i}{2\pi}\partial\bar{\partial}\log\det g_{k\bar{l}}\right]$
- Also have $c_i(V) = 0$ for $i > \mathrm{rank}(V), i > \dim_{\mathbb{C}}(M)$
- Chern classes encode topological information about bundle - in a sense they measure the extent of "non-triviality" of the bundle (trivial bundle is $E = M \times F$).

Calabi-Yau Mfolds Defn

CY mfold of real dimension $2m$ is a compact Kähler mfold (M, J, g) with

- zero Ricci form
- $c_1(M) = [0]$
- $\mathrm{Hol}(M) \subseteq SU(m)$ (Normally treat CY as those mfolds with $\mathrm{Hol}(M) = SU(m)$)
- trivial canonical bundle, $K_M = \wedge^m T^{*1,0} M$
- a globally defined, nowhere vanishing holomorphic m-form

These defns are all equivalent. Typically we use the second one.

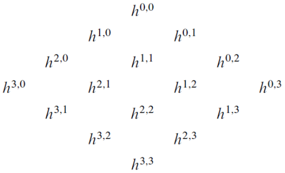
Calabi-Yau Proof (1)

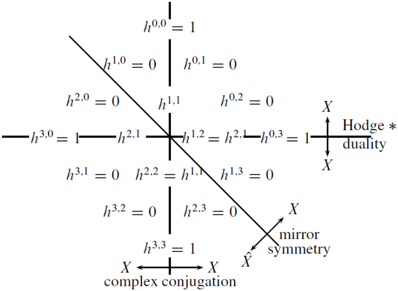
- Proof of equivalence of above defns can be found in hep-th/0702063
- Want to discuss equivalence of 1 and 2 - famously proved by Calabi and Yau.
- Since $c_1(M) = \left[\frac{1}{2\pi}\mathrm{Tr}\Theta\right]$ where Θ is curvature 2-form, it's obvious that zero Ricci form $\Rightarrow \Theta = 0 \Rightarrow c_1(M) = [0]$
- The converse (does a Kähler mfold with $c_1 = [0]$ admit a Ricci-flat metric?) is much more difficult to show.
- Calabi conjectured the answer was yes and managed to prove existence of Ricci-flat metric. Yau later proved uniqueness.
- Very complicated proof but highlights boil down to showing
    - If we take a cx mfold M with Kähler metric g and associated Kähler form ω.
    - $\exists$ a unique Ricci-flat metric g' (still Kähler) with associated Kähler form $\omega' \in [\omega]$
- This means there exists a unique Ricci-flat metric in each equivalence class of $H^{1,1}_{\bar{\partial}}(M)$. $[\omega] \in H^{1,1}_{\bar{\partial}}(M)$ are often called Kähler classes. Hence there exists a unique Ricci-flat Kähler metric associated with each Kähler class of M.

Calabi-Yau Proof (2)

- Since there is a unique Ricci-flat Kähler metric for each equivalence class in $H^{1,1}_{\bar{\partial}}(M)$ and there are $h^{1,1}$ many equivalence classes, the number of possible Ricci-flat metrics on our Calabi-Yau manifold is $h^{1,1}$.
- But wasn't the Ricci-flat metric postulated to be unique by Calabi? Why are there $h^{1,1}$ of them?
- The idea is that if you start with a Kähler mfold (M, J, g) then the Kähler form ω is in a fixed equivalence (Kähler) class and changing g cannot move you to another class. Consequently, we are in a fixed Kähler (equivalence) class $[\omega]$ of $H^{1,1}_{\bar{\partial}}(M)$ and then, as eluded to above, within each Kähler class, there is a unique Ricci-flat metric g' that's completely equivalent to the one we started with.

Symmetries of $h^{p,q}$ and Hodge Diamond for CY_3





Interested in CY_3. Hodge numbers $h^{p,q}$ run over $p, q = 0, \ldots, 3$ since elements of $H^{p,q}$ groups are closed (p, q)-forms, and these can't exceed the top form on the mfold - in this case a $(3, 3)$-form.
This gives Hodge Diamond.
Hodge Diamond has additional symmetries (apply in all dimensions). Shown explicitly here for CY_3 and explained below:

Hodge Duality and Complex Conjugation

- If $\omega \in A^{p,q}$ is harmonic then $\nabla_{\bar{\partial}}\omega = \nabla_{\partial}\omega = 0$[5]. Then we can check if $\bar{\omega} \in A^{q,p}$ is harmonic: $\nabla_{\partial}\bar{\omega} = \nabla_{\bar{\partial}}\bar{\omega} = 0$ by above. $\Rightarrow$ for each harmonic (p, q)-form, ω, $\exists$ harmonic (q, p)-form, $\bar{\omega}$. Since $H^{p,q} \simeq \mathcal{H}^{p,q}$, $h^{p,q}$ counts harmonic forms. Hence $h^{p,q} = h^{q,p}$ by **complex conjugation**.
- Take $[\omega] \in H^{p,q}, [\star\omega] \in H^{n-p,n-q}$, Then $\omega \wedge \star\omega$ is a top form (suitable volume element) and so $\int_M \omega \wedge \star\omega : H^{p,q} \times H^{n-p,n-q} \to \mathbb{C}$ is a non-singular map. This gives the following duality (isomorphism): $H^{p,q} \simeq H^{n-p,n-q}$ and hence $h^{p,q} = h^{n-p,n-q}$.

[5] $\nabla_d = \nabla_{\partial} = \nabla_{\bar{\partial}}$ on Kähler mfolds.

$h^{0,0}$ and $h^{n,n}$

- Because $H^{0,0} \simeq \mathcal{H}^{0,0}$, $h^{0,0}$ counts the dimension of the space of $(0,0)$-harmonic forms i.e. space of harmonic fns
- Harmonic fns satisfy max principle: On a compact space K, f achieves max/min on boundary. If K has no boundary then f must be constant.
- Since Calabi-Yau spaces are compact (see defn) and without boundary, harmonic fns must be const. This means $\mathcal{H}^{0,0} = \{\text{const}\} = \mathbb{C}$ and so $h^{0,0} = \dim_{\mathbb{C}} \mathcal{H}^{0,0} = 1$.
- This also fixes $h^{n.n} = 1$ by Hodge duality ($h^{p,q} = h^{n-p,n-q}$).

$h^{n,0}$ and $h^{0,n}$

- A trivial bundle is where the multiplet (E, B, π, F) satisfy $E = B \times F$.
- If a rank k vec bundle is trivial then $E = M \times \mathbb{C}^k$.
- CY mfolds have trivial canonical bundle, $K_M = \wedge^n T^{*1,0} M$
- Sections of K_M are $\propto dz^1 \wedge \cdots \wedge dz^n$. If we try to make another $(n, 0)$-form basis vector from the $\{dz^i\}$ we get something $\propto$ above (as it's top form). $\Rightarrow K_M$ is 1d[6]
- Trivial canonical bundle $\Rightarrow E = M \times \mathbb{C}$.
- Corresponding to $M \times \{1\}$, there is a particular holomorphic $(n, 0)$-form, Ω, called *holomorphic volume form*. So trivial $K_M \Rightarrow \exists$ at least 1 $(n, 0)$-form.
- Ω is holomorphic and hence $\bar{\partial}\Omega = 0 \Rightarrow [\Omega] \in H^{n,0}$.
- We have seen that other such $(n, 0)$-forms are $\propto \Omega$ i.e. of the form $f\Omega$ where f is some cx fn.
- Maximum modulus principle says f holomorphic $\Rightarrow f =$ const. Because $f =$ const, we have $\Omega \sim f\Omega$ and thus $H^{n,0} = \{[f\Omega]\}$ and so $h^{n,0} = \dim_{\mathbb{C}} H^{n,0} = 1$
- This fixes $h^{0,n} = 1$ by complex conjugation $(h^{p,q} = h^{q,p})$.

[6] Only has 1 basis vector: $dz^1 \wedge \cdots \wedge dz^n$

Brief Aside: Anti-Canonical Bundle

- If M is CY mfold with canonical bundle $K_M = \wedge^n T^{*1,0}M$. then the anticanonical bundle K_M^* is defined as the bundle, whose Whitney sum[7] with the canonical bundle is the trivial bundle.
- For a CY mfold, K_M is already a trivial bundle hence the anticanonical bundle is empty.
- The first Chern class of M is the same as the first Chern class of K_M^* i.e. $c_1(M) = c_1(K_M^*) = -c_1(K_M)$.
- Since K_M^* is empty, $c_1(K_M^*) = 0$ and so $c_1(M) = 0$ as it should be for a CY mfold.
- This is just an aside to tie together some of the defns of a CY mfold.

[7] means for adding bundles - see Nakahara

Summary

For our CY_3, we now have $h^{0,0} = h^{3,3} = h^{0,3} = h^{3,0} = 1$.

To make the following arguments simpler, we will restrict to CY_3 although they can be generalised to cover CY_n as well.

$h^{p,0}$ for $0 < p < n = 3$

- Theorem[8] says that for any d-**harmonic** s-form ζ, let $F(\zeta) := R_m{}^n \zeta_{[nr_2 \dots r_s]} \zeta^{[mr_2 \dots r_s]} + \frac{s-1}{2} R_m{}^n{}_p{}^q \zeta_{[nqr_3 \dots r_s]} \zeta^{[mpr_3 \dots r_s]}$ and if $F(\zeta)$ is positive semi-definite then ζ is covariantly constant.
- We will look for harmonic 1-forms so we take $s = 1$. This kills the 2nd term of F. The 1st term vanishes because CY mfolds are Ricci-flat ($R_m{}^n = 0$). This means that $F(\zeta) = 0$ i.e. F is positive semi-definite for 1-forms on CY mfolds.
- To complete the theorem and find a harmonic 1-form, it remains to show ζ is covariantly constant i.e. doesn't change under parallel transport.
- However, we know that CY_3 mfolds have a $SU(3)$ holonomy group and a 1-form ζ will transform under the $3 \oplus \bar{3}$ rep of $SU(3)$ i.e. is changed by parallel transport.
- $\Rightarrow \zeta$ not covariantly constant $\Rightarrow \zeta$ not harmonic $\Rightarrow \nexists$ d-harmonic 1-form.
- d-harmonic refers to de Rahm cohomology so $\mathcal{H}^1 = \emptyset$ and since $H^1 \simeq \mathcal{H}^1$, $b^1 = 0$.
- But $b^1 = \displaystyle\sum_{p+q=1} h^{p,q} = h^{0,1} + h^{1,0}$ and since $h^{p,q} \geq 0$, we must have $h^{0,1} = h^{1,0} = 0$

[8] p32 CY Mfolds: A Bestiary for Physicists, T. Hübsch

Killing The Remaining Peripheral Hodge Numbers

- A CY_n is known to have a unique holomorphic $(n, 0)$-form Ω (holomorphic volume form). This is a $(3, 0)$-form for CY_3.
- If we take $[\alpha] \in H^{0,q}, \exists$ **unique** $[\beta] \in H^{0,3-q}$ such that $\int_M \alpha \wedge \beta \wedge \Omega = 1$ (N.B. integrand is a $(3, 3)$ top-form).
- This sets up a duality (isomorphism) between $H^{0,q}$ and $H^{0,3-q}$.
- Hence $h^{0,q} = h^{0,3-q}$ (sometimes called holomorphic duality).
- We had $h^{0,1} = 0$. So $h^{0,2} = 0$ by holo duality. Then $h^{2,0} = 0$ by conjugation. Then $h^{1,3} = 0$ by Hodge dual and then $h^{3,1} = 0$ by conjugation.
- We also had $h^{1,0} = 0 \Rightarrow h^{2,3} = 0$ by Hodge duality. Then $h^{3,2} = 0$ by conjugation.
- To summarise $h^{1,0} = h^{0,1} = h^{2,0} = h^{0,2} = h^{2,1} = h^{1,2} = h^{3,1} = h^{1,3} = 0$
- The outside of Hodge diamond are fixed as 1s or 0s (true for CY_n not just CY_3).

Independent Hodge Numbers on CY_3

- Remaining unfixed Hodge numbers in CY_3 diamond are $h^{1,1}, h^{1,2}, h^{2,1}, h^{2,2}$.
- These are not independent as $h^{1,2} = h^{2,1}$ (conjugation) and $h^{1,1} = h^{2,2}$ (Hodge dual).
- Independent Hodge numbers for CY_3 are $h^{1,1}$ and $h^{2,1}$.
- $h^{1,1}$ measures deformations of Kähler structure (ω) and $h^{2,1}$ measures deformations of complex structure (J).
- Since $h^{p,q} \geq 0 \quad \forall p, q$, we know $h^{2,1} \geq 0$ and $h^{1,1} \geq 1$ (CY is Kähler so $\exists$ at least one $\bar{\partial}$-closed $(1,1)$-form: ω)
- **Important Note:** In higher dimensions, there are obviously more independent Hodge numbers. e.g. for $d = 4$, 3 are independent.

Mirror Symmetry

- There is a fascinating symmetry of CY mfolds, called *mirror symmetry*, that can be seen on Hodge Diamond.
- Given a CY mfold $M, \exists$ another CY mfold M' of same dimension s.t. $h^{p,q}(M) = h^{3-p,q}(M')$.
- This mirror symmetry exchanges $h^{1,1}$ and $h^{2,1}$ on Hodge diamond.
- Although two CY mfolds M, M' may look very different geometrically, string theory compactification on these manifolds leads to **identical** effective field theories.
- **Means that CY mfolds come exist in mirror pairs** (M, M').
- IIA on M mirror dual to IIB on M' whilst IIB on M mirror dual to IIA on M'.
- Mirror symmetry can be shown to be special case of T-duality (Strominger, Yau, Zaslow).
- Interesting case when we consider the mirror dual of CY mfold M with $h^{2,1}(M) = 0$. Can anyone see the problem? The mirror dual M' will have $h^{1,1}(M') = 0$. However, since CY mfolds are a subclass of Kähler mfolds (for which $\exists$ fundamental $(1,1)$-form) which all have $h^{1,1} \geq 1$, the mirror dual M' is not CY, or even Kähler. $\Rightarrow \exists$ an extended space of compact spaces (see Wiki article).
- A CY mfold with $h^{2,1} = 0$ is called a **rigid Calabi-Yau**.

What can we say about M in this extended space?

- This was mentioned in talk by Keshav Dasgupta on 8/5/13.
- He discussed how mirror symmetries could talk Kähler mfolds to non-Kähler mfolds (Kähler mfolds all have $h^{1,1} \geq 1$ so will be similar to CY case discussed on previous slide.
- **What are these other manifolds? Can we name/describe them?**
- They are things like *"balanced mfolds"*, *"half-flat mfolds" etc*
- But what are these? There is a paper "Non-Kähler String Backgrounds and their Five Torsion Classes" by Cardoso *et al.* that discusses this: **[arXiv: 0211118v3]**

Torsion Classes: Mfold Classification for Physicists

- If we are on a 6d mfold (3 cx dimensions) e.g. CY_3 then we can define 5 so-called $SU(3)$ structures or torsion classes $W_1, \ldots, W_5$.
- If we know the fundamental $(1,1)$-form, ω, and the holomorphic $(3,0)$-form, Ω, then it is possible to calculate all of $W_1, \ldots, W_5$ very easily (see Cardoso paper for exact formulae).
- This provides a **much simpler** way of classifying manifolds as CY, Kähler etc than e.g. computing Chern classes. Interestingly, mathematicians already knew about these structures before CY mfolds were studied in physics.
- Some interesting relations are:
  - $W_1 = W_2$ then mfold is hermitian (and hence also complex).
  - $2W_4 = W_5$ then mfold preserves SUSY
  - $W_1 = \cdots = W_5 = 0$ then mfold is CY_3
- There are a variety of other relations on the W_i that allow us to classify non-CY mirror mfolds (which arise if the original CY_3 mfold has $h^{2,1} = 0$) or indeed allow us to classify manifolds in general.

Alternative View: SCFTs (1)

- See arXiv: hep-th/9304045 for details
- Better statement of mirror symmetry is that CY mfolds are the realisation of an $\mathcal{N} = 2$ SCFT.
- A given SCFT can be realised as a CY mfold in two different ways: M and M' (whose Hodge numbers are related by mirror symmetry).
- In underlying SCFT there's no natural way to decide which operators correspond to $(1, 1)$-forms and which correspond to $(2, 1)$-forms in an associated CY.
- Because we don't know which type of forms to assign the operators to, someone created mirror symmetry in which any SCFT corresponds to a pair of CYs where the role of these two types of forms are exchanged.

Alternative View: SCFTs (2)

- How does the SCFT viewpoint handle the mirror of a rigid Calabi-Yau?
- Paper by Candelas et al. considers a specific rigid Calabi-Yau, M, with $h^{2,1} = 0, h^{1,1} = 36$.
- By identifying M with the Gepner model 1^9, it's possible to give a geometric interpretation to M' as a representative of a class of generalised Calabi-Yau mfolds of dimension 7 with positive first Chern class.
- Despite having odd dimensions, these generalised CYs correspond to SCFTs with $c = 9$ and so are perfectly good for compactifying heterotic string to 4 dimensions of spacetime.
- As a final note on mirror symmetry, we point out that it is still poorly understood. In particular,
    - It hasn't been proven (it's not known under what circumstances mirror symmetry is true).
    - There's no general procedure for constructing the mirror of a given CY mfold. Only a few examples are explicitly known.

What's the point? Why is Hodge diamond important?

- Euler characteristic $\chi = 2(h^{1,1} - h^{2,1})$
- Earlier we said we were interested in CY 3-folds for dimensional reasons and now, because we want 3 generation models (with $\chi = \pm 6$) we can further restrict to only CY 3-folds with $h^{1,1} - h^{2,1} = \pm 3$.
- It may be tricky to compute $h^{1,1}, h^{2,1}$ for certain CY 3-folds. However, there are many ways of computing χ. Often it's easier to find χ and one of the Hodge numbers. This then fixes the other and once we have $(\chi, h^{1,1}, h^{2,1})$, all topological info for the CY_3 is fixed.

Constructions: An Example (1)

- So far, all extremely abstract. Try to conclude with a more tangible example.
- Recall that the weighted projective $\mathbb{WP}^4_{a,b,c,d,e}$ satisfies $(v, w, x, y, z) \sim (\lambda^a v, \lambda^b w, \lambda^c x, \lambda^d y, \lambda^e z)$ where a, b, c, d, e are the weights of each of the coordinates v, w, x, y, z respectively. The equivalence relation with the weights means $\mathbb{WP}^4$ is a 4d surface in $\mathbb{C}^5$ (hence 5 coordinates with 1 constraint).
- Now complex projective space is a type of weighted projective space where all the weights are 1. So, we see $\mathbb{CP}^4 = \mathbb{WP}^4_{1,1,1,1,1}$
- We claim that whenever we have $\mathbb{CP}^n$, then a polynomial of order $n + 1$ (which means each term has weight $n + 1$) is a Calabi-Yau mfold of dimension $n - 1$
- Thus, if we look at $\mathbb{CP}^4$, a polynomial where each term is of weight 5 is an example of a Calabi-Yau 3-fold.

Constructions: An Example (2)

- An example would be the quintic polynomial $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$ in $\mathbb{CP}^4$. Since in complex projective spaces, all weights are 1, each term here is clearly of weight $4 + 1 = 5$. Thus this quintic should define a CY mfold.
- Recall that $\mathbb{CP}^4$ is space of lines in $\mathbb{C}^5$ (hence 5 coordinates $z_1, \ldots, z_5$). But the weighted equivalence relation defines a 4d surface within $\mathbb{C}^5$ so $\mathbb{CP}^4$ is really 4d. Then the polynomial eqn reduces this further to a 3d surface - thus CY mfold is a 3d hypersurface within the 4d space $\mathbb{CP}^4$.
- Thus the quintic in $\mathbb{CP}^4$ really defines a CY 3-fold (which is what we're interested in as string theorists).
- It's the most famous and widely studied example of a CY_3.

Constructions: An Example (3)

- Can we check quintic in $\mathbb{CP}^4$ is really a CY mfold?
- The "hands-on" way to check something is CY is to find a globally defined, nowhere vanishing holomorphic $(n, 0)$ form.
- We'll give sketch of how to prove it using Chern classes since we discussed them above.
- To begin with we need to examine the Chern classes of $\mathbb{CP}^n$

Constructions: An Example (4)

- Homogeneous coordinates z^i are coordinates in $\mathbb{C}^{n+1}$ before we identify points on same line to form $\mathbb{CP}^n$ (this is the weighted identification $(z^0, \ldots, z^n) \sim (\lambda z^0, \ldots, \lambda z^n)$).
- $\Rightarrow \frac{\partial}{\partial z^i}$ are basis vectors on $T^{(1,0)}\mathbb{C}^{n+1}$.
- If L is *tautological line bundle* (line bundle whose fiber, F, is the line it respresents in $\mathbb{C}^{n+1}$) then the *hyperplane line bundle* L^{-1} is the dual line bundle we must add (Whitney sum) to L to get the trivial bundle.
- If $s_i(z)$ are sections of L^{-1}, we can view them as fns/coordinates on $\mathbb{C}^{n+1}$
- $\Rightarrow T^{(1,0)}\mathbb{C}^{n+1}$ is spanned by $s_i(z)\frac{\partial}{\partial z^i}$
- As for $\mathbb{CP}^n$, $T^{(1,0)}\mathbb{CP}^n$ is also spanned by $s_i(z)\frac{\partial}{\partial z^i}$ (since $\frac{\partial}{\partial z^i}$ are $(n+1)$-dimensional, they're guaranteed to span (cover) the n-dimensional space $\mathbb{CP}^n$).

Constructions: An Example (5)

- Now let's call the hyperplane line bundle $L^{-1} = \mathcal{O}_{\mathbb{CP}^n}(1)$ following Bouchard's notation.
- Its sections are $s_i(z)$ (see last slide)
- Thus $\exists$ a map $f : \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \to T^{(1,0)}\mathbb{CP}^n$ (i.e. acts on $n+1$ copies of hyperplane line bundle) such that Ker(f) is the trivial line bundle $\mathbb{C}$
- i.e. $f : (z_0, \ldots, z_n) \mapsto z_i \frac{\partial}{\partial z_i} \simeq 0$ in $\mathbb{CP}^n$
  N.B. $z_i \frac{\partial}{\partial z_i} \simeq 0$ in $\mathbb{CP}^n$ whilst $s_i(z) \frac{\partial}{\partial z_i} \not\simeq 0$ in $\mathbb{CP}^n$ i.e. $\mathrm{Ker}(f) = \{s_i(z) | s_i(z) = z_i\}$
- We can summarise this with the following short, exact sequence:
  $0 \to \mathbb{C} \to \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \to T^{(1,0)}\mathbb{CP}^n \to 0$

Constructions: An Example (6)

- By exact sequence, we mean that the image of one map is the kernel of the next map c.f. nilpotency in cohomology. We can do some checks of this:
- Im $(0 \to \mathbb{C}) = \{0\} \Rightarrow$ Ker $(\mathbb{C} \to \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}) = \{0\}$ This means $\mathbb{C} \to \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}$ is injective.
- Ker $(T^{(1,0)}\mathbb{CP}^n \to 0) = T^{(1,0)}\mathbb{CP}^n \Rightarrow$ Im $({\mathcal{O}_{\mathbb{CP}^n(1)}}^{\oplus(n+1)} \to T^{(1,0)}\mathbb{CP}^n) = T^{(1,0)}\mathbb{CP}^n$ This means ${\mathcal{O}_{\mathbb{CP}^n(1)}}^{\oplus(n+1)} \to T^{(1,0)}\mathbb{CP}^n$ is surjective.
- A property of Chern classes says that if we have a short exact sequence $0 \to A \to B \to C \to 0$ then $c(A) = \frac{c(B)}{c(C)}$ where $c(A)$ is the total Chern class of A
- Thus $c(\mathbb{C}) = \frac{c(\mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)})}{c(T^{(1,0)}\mathbb{CP}^n)}$

Constructions: An Example (7)

- But $c(\mathbb{C}) = 1$ trivially (since $\mathbb{C}$ is flat so Ricci 2-form vanishes)
- Thus $c(\mathbb{CP}^n) := c(T^{(1,0)}\mathbb{CP}^n) = \frac{c(\mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)})}{c(\mathbb{C})} = c(\mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}) = [c(\mathcal{O}_{\mathbb{CP}^n}(1))]^{\oplus(n+1)}$
- But $\mathcal{O}_{\mathbb{CP}^n}(1)$ is a line bundle $\Rightarrow c(\mathcal{O}_{\mathbb{CP}^n}(1)) = 1 + c_1(\mathcal{O}_{\mathbb{CP}^n}(1)) + 0$ since $c_i(V) = 0 \quad \forall \quad i > \text{rank }(V) = 1$ (line bundle)
- Let $x = c_1(\mathcal{O}_{\mathbb{CP}^n}(1))$.
- Then $c(\mathbb{CP}^n) = (1 + x)^{n+1}$

Constructions: An Example (8)

- Next we consider a hypersurface X in $\mathbb{CP}^n$ defined by zero locus of some polynomial of order d e.g. our CY_3 in $\mathbb{CP}^4$ (for which $d = 5$).
- N_X is the normal bundle of X (consisting of vectors normal to hypersurface X - see picture in Bouchard's notes).
- N_X defined by $N_X = \frac{T^{(1,0)}\mathbb{CP}^n|_X}{T^{(1,0)}X}$ (quotient space)
- This means we form N_X by taking all holomorphic tangent vectors in $\mathbb{CP}^n$ restricted to hypersurface X and any that are tangent to X are identified. Since vectors on X can be decomposed into tangent and normal parts, this leaves just the normal vectors.
- If we take the Chern class of the above quotient space we get $c(N_X) = c\left(\frac{T^{(1,0)}\mathbb{CP}^n|_X}{T^{(1,0)}X}\right) = \frac{c\left(T^{(1,0)}\mathbb{CP}^n|_X\right)}{c\left(T^{(1,0)}X\right)}$
- Earlier we say that a short exact sequence between A, B and C set up a relationship between the Chern classes. We can now reverse this to get the following short exact sequence $0 \to T^{(1,0)}X \to T^{(1,0)}\mathbb{CP}^n|_X \to N_X \to 0$ for which we could check exactness (injectivity/surjectivity) as before.

Constructions: An Example (9)

- So we have $c(X) := c(T^{(1,0)}X) = \frac{c\left(T^{(1,0)}\mathbb{CP}^n|_X\right)}{c(N_X)}$
- But sections of hyperplane line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$ are the coordinates z^i (Bouchard).
- Hypersurface X is defined by zero locus of some polynomial of order d in the coordinates z^i. But if we use locus of values $1, 2, 3, \ldots$ then this just picks a different hypersurface in $\mathbb{CP}^n$ i.e. shifts surface up/down normal bundle N_X
- $\Rightarrow N_X = \mathcal{O}_{\mathbb{CP}^n}(d)$ i.e. normal bundle is hyperplane line bundle of order d (importantly it is **still** a **line** bundle so $c_i = 0 \forall i > 1$
- Thus $c(N_X) = c(\mathcal{O}_{\mathbb{CP}^n}(d)) = 1 + c_1(\mathcal{O}_{\mathbb{CP}^n}(d))$ (expansion terminates at c_1 since it's line bundle so $c_i = 0 \forall i > 1$)

Constructions: An Example (10)

- Earlier we had $c(\mathcal{O}_{\mathbb{CP}^n}(1)) = 1 + x$
- Since $c(M) = \det\left(1 + \frac{it}{2\pi}\Theta\right)$ and $ch(M) = \mathrm{tr}\left(\exp\left(\frac{it}{2\pi}\Theta\right)\right)$. Thus $ch(\mathcal{O}_{\mathbb{CP}^n}(1)) = e^x$
- $\Rightarrow ch(\mathcal{O}_{\mathbb{CP}^n}(d)) = e^{dx} \Rightarrow c(\mathcal{O}_{\mathbb{CP}^n}(d)) = 1 + dx$ (no higher terms since it's a line bundle)
- On previous slide we had $c(X) = \frac{c(T^{(1,0)}\mathbb{CP}^n|_X)}{c(N_X)} = \frac{c(\mathbb{CP}^n)}{c(\mathcal{O}_{\mathbb{CP}^n}(d))} = \frac{(1+x)^{n+1}}{1+dx}$ by substituting earlier results.
- But $(1 + dx)^{-1} = 1 - dx + \mathcal{O}(x^2) = 1 - dx$ as higher order terms vanish for line bundle
- And $(1 + x)^{n+1} = 1 + (n + 1)x + \mathcal{O}(x^2) = 1 + (n + 1)x$ with higher order terms vanishing for same reason
- Thus $c(X) = (1+x)^{n+1}(1+dx)^{-1} = (1 + (n + 1)x)(1 - dx) = 1+(n+1)x-dx+\mathcal{O}(x^2)$ with higher order terms vanishing again
- This simplifies to $c(X) = \underbrace{1}_{c_0(X)} + \underbrace{(n + 1 - d)x}_{c_1(X)}$

Constructions: An Example (11)

- Thus if the polynomial is of degree $d = n + 1$, then $c_1(X) = 0$ and X is a CY mfold.
- For our case of the hypersurface in $\mathbb{CP}^4$ defined by the zero locus of the quintic polynomial $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$, we have $d = 5$ and $n = 4$.
- Clearly $n + 1 - d = 0$ so the first Chern class vanishes and the hypersurface clearly defines a CY_3.