MORSE FUNCTIONS FOR WHICH THE STATIONARY PHASE APPROXIMATION IS EXACT

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Let a nondegenerate Morse function be given by \( f: X \to \mathbb{R} \) with isolated critical points and distinct critical values on a compact \( n \)-dimensional manifold \( X \), with a given volume form \( dm \). The method of stationary phase gives one an approximation to the integral \( \int_X e^{if} \, dm \) when \( t \) is large in terms of the values of \( f \) and its derivatives at the critical points. In fact for each critical point \( x \) there is a power series

\[
A_x = \sum_{j \geq 0} a_{x,j} t^{-j},
\]

where the \( a_{x,j} \) are real constants depending on the values of \( f \) and its derivatives at \( x \), with the following property. If \( A_{x,k}(t) \) denotes the value at \( t > 0 \) of the \( k \)th partial sum of \( A_x \) and \( \lambda(x) \) denotes the index of \( f \) at \( x \) then the sum

\[
\sum_{df(x) = 0} \left( \frac{2\pi}{t} \right)^{n/2} e^{i\lambda(x)} e^{(n - 2\lambda(x))x_i/4} A_{x,k}(t)
\]

approximates \( \int_X e^{if} \, dm \) to order \( (n/2) + k \) in \( t^{-1} \) as \( t \to \infty \). For the precise values of the constants \( a_{x,j} \) see [5, 7.7.5].

Let the stationary phase approximation be called exact if the power series \( A_x \) converge for all critical points \( x \) and all sufficiently large \( t > 0 \) and the numbers \( A_n(t) \) to which they converge satisfy

\[
\int_X e^{if} \, dm = \left( \frac{2\pi}{t} \right)^{n/2} \sum_{df(x) = 0} e^{i\lambda(x)} e^{(n - 2\lambda(x))x_i/4} A_x(t).
\]

The purpose of this note is to show that this can only happen for very special Morse functions.

**Theorem.** If the stationary phase approximation for \( f \) is exact then the index of every critical point is even. Hence \( f \) is a perfect Morse function (i.e. its Morse inequalities are in fact equalities).

This theorem has the following immediate consequence.

**Corollary.** If the stationary phase approximation for \( f \) is exact then the homology of \( X \) is torsion-free and occurs only in even dimensions. In particular the dimension of \( X \) is even.
The exactness of the stationary phase approximation has been of interest recently because of a result of Duistermaat and Heckman (see [3] and also [1]). They have shown that if $X$ is a compact symplectic manifold with symplectic form $\omega$ and if $f$ is a component of a momentum map for a compact Lie group action on $X$ then the stationary phase approximation for $\int_X e^{itf} \omega^n/n!$ is exact. Indeed the asymptotic expansion terminates after the first term and gives the value of $\int_X e^{itf} \omega$ for all $t \neq 0$.

**Proof of the theorem.** Since $X$ is compact, by differentiating under the integral sign one sees that the function $g: \mathbb{C} \to \mathbb{C}$ defined by

$$g(t) = \int_X e^{itf} \omega$$

is holomorphic. Assume that for some $t_0 > 0$ the power series $A_x$ converge for all real $t > t_0$. Hence they converge absolutely for all complex $t$ such that $|t| > t_0$, and thus give holomorphic functions in the annulus defined by $|t| > t_0$. Therefore the function

$$a(t) = \sum_{d/(x) = 0} e^{itf(x)} e^{(n - 2\lambda(x)) \pi i / 4} A_x(t)$$

is also holomorphic there. By assumption, $g(t) = (2\pi/t)^{n/2} a(t)$ when $t$ is real and $t > t_0$. Suppose that $a(t)$ does not vanish identically. Since non-zero holomorphic functions on $\mathbb{C}$ have isolated zeros and satisfy the principle of analytic continuation, it follows that $t^{n/2}$ extends to a single-valued meromorphic function of $t$ in the annulus, so that $n$ is even, and moreover $g(t) = (2\pi/t)^{n/2} a(t)$ whenever $|t| > t_0$.

Note that $g(t)$ is the complex conjugate of $g(-t)$ for all real $t$. Hence for all real $t > t_0$ $(2\pi/t)^{n/2} a(t)$ is the complex conjugate of $(-2\pi/t)^{n/2} a(-t)$; that is,

$$a(t) = (-1)^{n/2} a(-t)$$

for all real $t$ with $|t| > t_0$. Of course this is also true if $a(t)$ vanishes identically.

Since each $A_x$ is a power series in $(it)^{-1}$ with real coefficients,

$$A_x(t) = \overline{A_x(-t)}$$

for all real $t$ such that $|t| > t_0$. Therefore by (1)

$$(-1)^{n/2} a(t) = (-1)^{n/2} \sum_{d/(x) = 0} e^{itf(x)} e^{-(n - 2\lambda(x)) \pi i / 4} A_x(t) = \sum_{d/(x) = 0} (-1)^{\lambda(x)} e^{itf(x)} e^{(n - 2\lambda(x)) \pi i / 4} A_x(t)$$

whereas

$$a(t) = \sum_{d/(x) = 0} e^{itf(x)} e^{(n - 2\lambda(x)) \pi i / 4} A_x(t).$$

Hence from (2)

$$\sum_{d/(x) = 0, \lambda(x) \text{ odd}} e^{itf(x)} (-1)^{\lambda(x)} A_x(t) = 0.$$
whenever \(|t| > t_0\). But when \(|t| > t_0\) each \(A_x(t)\) is an absolutely convergent power series in \((it)^{-1}\) so \(A_x(t)\) converges to \(a_{x,0}\) as \(t \to \infty\). Therefore
\[
h(t) = \sum_{d(f(x))=0, j(x) \text{ odd}} e^{itf(x)} (-j(x)) a_{x,0}
\]
tends to 0 as \(t \to \infty\). Hence \(h(t)\) is a bounded holomorphic function on the complex plane, so it must be constant. Thus \(h(t) = 0\) for all \(t \in \mathbb{C}\), but by assumption the values of \(f\) at its critical points are distinct, and hence the functions \(\{e^{itf(x)}|df(x) = 0\}\) are linearly independent over \(\mathbb{C}\).

It follows that \(a_{x,0} = 0\) whenever \(df(x) = 0\) and \(\lambda(x)\) is odd. However
\[
a_{x,0} = |\det H_f(x)|^{-1/2},
\]
where \(H_f(x)\) is the Hessian of \(f\) at \(x\), regarded as a non-degenerate endomorphism of the tangent space \(T_x X\) via some Riemannian metric on \(X\), which induces the given volume form (see e.g. [4, p. 61 or [5, 7.7.5]). In particular \(a_{x,0} > 0\) for all critical points \(x\). Therefore \(\lambda(x)\) is even for every critical point \(x\).

The Morse inequalities of \(f\) with respect to any field of coefficients \(K\) can be expressed in the form
\[
\sum_{d(f(x))=0} t^{\lambda(x)} - P_t(X; K) = (1+t) Q_k(t),
\]
where \(P_t(X; K) = \sum_{i \in \mathbb{N}} t^i \dim_K H^i(X; K)\) and \(Q_k(t) \in \mathbb{Z}[t]\) satisfies \(Q_k(t) \geq 0\) (that is, all the coefficients of \(Q_k(t)\) are non-negative). It is easy to see from the form of these inequalities that if every \(\lambda(x)\) is even then \(Q_k(t) = 0\). (This is a special case of the lacunary principle.) This says precisely that the Morse inequalities are equalities, or equivalently that \(f\) is a perfect Morse function. Therefore the proof of the theorem is complete.

Remarks. (1) A slightly more general form of the method of stationary phase yields an asymptotic expansion for \(\int_X \phi e^{itf} dm\), where \(\phi: X \to \mathbb{R}\) is any smooth function. Its first term gives
\[
\int_X \phi e^{itf} dm = \left(\frac{2\pi}{i}\right)^n \sum_{j(x)=0} \frac{e^{i(n-2\lambda(x))x/4} e^{itf(x)} \phi(x)}{|\det H_f(x)|} + O(t^{-(n+1)})
\]
as \(t \to \infty\) [5, 7.7.5]. The argument used to prove the theorem also shows that if this form of the stationary phase approximation is exact for some \(\phi\) which does not vanish at any critical point of \(f\), then every critical point has even index and \(f\) is a perfect Morse function. This is equivalent to saying that the volume form \(dm\) may be replaced by any \(n\)-form on \(X\) which does not vanish at the critical points of \(f\).

(2) The proof that if the stationary phase approximation for \(f\) is exact then every index is even generalises immediately to the case when \(f\) is a non-degenerate Morse function in the sense of [2]. Then the set of critical points is a finite disjoint union of connected orientable submanifolds of \(X\): let these be \(C_1, \ldots, C_p\). The method of stationary phase gives an asymptotic expansion for \(\int_X e^{itf} dm\) of the form
\[
\left(\frac{2\pi}{i}\right)^{n/2} \sum_{1 \leq j \leq p} e^{i\lambda(C_j)/2} e^{itf(C_j)} \sum_{k \leq 0} a_{jk} |it|^k,
\]
where \(\lambda(C_j)\) is the index of \(f\) along \(C_j\) (see [5, 7.7.6] or generalise the proof of [4] in the obvious way using the Morse lemma with parameters). The Morse inequalities for \(f\) now take
the form
\[ \sum_{1 \leq j \leq p} t^{i(C_j)} P_t(C_j; K) - P_t(X; K) = (1 + t) Q_t(K), \]
where \( Q_t(K) \geq 0 \). Unfortunately the lacunary principle no longer applies: it is not possible in general to deduce that \( f \) is a perfect Morse function from the fact that every index is even.

(3) In [6] Witten gives a new proof of the Morse inequalities and a description of what happens when they are not equalities in terms of tunnelling effects. It is conceivable that his methods might give a more illuminating, if less elementary, proof that any Morse function with an exact stationary phase approximation is perfect. Such a proof might also generalise to Morse functions without isolated critical points.

(4) In order to prove the theorem (and its generalisation described in remark (1)) it was assumed that the critical values of \( f \) were distinct. If this is not the case the proof gives for each critical value \( x \) an equality
\[ \sum_{x \neq x_0 \text{ mod } 4} \phi(x) \alpha_{\nu, 0} - \sum_{x = x_0 \text{ mod } 4} \phi(x) \alpha_{\nu, 0} = 0, \]
where \( \alpha_{\nu, 0} > 0 \) for all critical \( x \). If \( \phi \) is positive this is still enough to show that the dimension of \( X \) is even: otherwise choosing \( x \) to be the maximum value taken by \( f \) gives
\[ \sum_{x = x_0} \phi(x) \alpha_{\nu, 0} = 0, \]
which is impossible since the left hand side is strictly positive.

On the other hand if \( \phi \) is allowed to take negative values then there exist counterexamples to the theorem in all dimensions. It suffices to find a counter-example in dimension 1: then one can take products. Let
\[ X = S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \]
with the standard metric, and let \( f \) and \( \phi \) be the restrictions of \( x^2 \) and \( x + y \) respectively. Then \( f \) is not a perfect Morse function but \( \int_X \phi e^{itf} dm \) is zero for all \( t \), and hence must equal its asymptotic expansion.

It is unclear whether counter-examples exist with \( \phi \) positive and the dimension of \( X \) even and at least 4.

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REFERENCES

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