The action for self-dual p-form gauge fields

and the geometry of gravitons
Half Field Theory

- $q - 1$-form gauge field $A$, $F = dA$

\[ A_{\mu_1...\mu_{q-1}} \quad \text{Antisymetric tensor gauge field} \]

\[ F_{\mu_1...\mu_q} = (q + 1)\partial_{[\mu_q} A_{\mu_1...\mu_{q-1}]} \quad \text{Field strength} \]
• $q - 1$-form gauge field $A$, \( F = dA \)

• If $d = 2q$, and $q$ odd: can impose SELF-DUALITY \( F = * F \) halves d.o.f.

• Covariant action for SD theory?

\[
(*F)_{\mu_1 \ldots \mu_q} = \frac{1}{q!} \sqrt{|g|} e_{\mu_1 \ldots \mu_q \nu_1 \ldots \nu_q} g^{\nu_1 \rho_1} \ldots g^{\nu_q \rho_q} F_{\rho_1 \ldots \rho_q}
\]

\[
dF = 0, \quad *F = F \implies d*F = 0 \quad \text{Field equation}
\]
Half Field Theory

- $q$ - 1-form gauge field $A$, $F = dA$
- If $d = 2q$, and $q$ odd: can impose SELF-DUALITY $F = *F$ halves d.o.f.
- Covariant action for SD theory?
- Sen's action: inspired by String Field Theory
- Quadratic: good for quantisation
- Generalises to allow Born-Infeld and Chern-Simons interactions
Sen’s Theory

- Spacetime metric $g_{\mu\nu}$, Minkowski metric $\eta_{\mu\nu}$, Hodge duals $\ast = \ast_g, \ast_\eta$

- Fields in action couple to $\eta$, there is weird interaction term depending on $g_{\mu\nu}$

- TWO SD gauge fields $A, C$ (constructed from fields appearing in action)

  $$ F = dA, F = \ast F \quad \quad G = dC, G = \ast_\eta G $$

- $A$: couples to space-time metric (and other physical fields)

- $C$: Couples to none of the physical fields: DECOUPLES
Sen’s Theory

- $\eta_{\mu\nu}$ very restrictive: Most spacetimes don’t admit Minkowski metric
- Coordinate independent?
- Strange symmetry: acts like diffeomorphisms on $g_{\mu\nu}$, $A$ but $\eta_{\mu\nu}$, $C$ invariant
- Hard to work with explicitly — given by infinite sum of interactions
- Would like coordinate independent theory that can be formulated on any spacetime
  e.g. AdS. Action for IIB on $AdS_5 \times S^5$?
Non-Sen’s Theory

- Replace $\eta_{\mu\nu}$ with metric $\bar{g}_{\mu\nu}$. $\eta \rightarrow \ast \bar{g} = \ast$

- $F = dA, F = \ast F, \quad G = dC, G = \ast G$

- Hard bit: finding interaction term $f(g, \bar{g})$ and showing it gives required field equations

- Physical Sector $g_{\mu\nu}, A +$ other physical fields, couple to each other

- Non-physical sector $\bar{g}_{\mu\nu}, C$ couple to each other but not to any physical fields

- Gives desired physical sector plus shadow sector that decouples

- Closed form, not infinite sum. This helps in quantum calculations
The space with 2 metrics

- Spacetime with 2 metrics $\mathcal{M}(g, \tilde{g})$
- Interesting bi-metric geometry, new structures, important in technical bits
- Action covariant, can be formulated on any spacetime
- $\tilde{g}_{\mu\nu}$ can be a background metric or can be dynamical
- Similar “bi-metric structures” arise in massive gravity and interacting theory of 2 gravitons; de Rham, Gabadadze, Tolley; Hassan, Rosen
Doubled Geometry (after all)

- Two metrics: 2 kinds of “diffeomorphism” symmetries
  \[ \delta g_{\mu\nu} = 2\partial_{(\mu}\zeta_{\nu)} + \ldots, \quad \delta \bar{g}_{\mu\nu} = 2\partial_{(\mu}\chi_{\nu)} + \ldots \]
- Extends to symmetries of full theory
- The \( \zeta_\mu \) transformations act on Physical Sector \( g_{\mu\nu}, A \) + other physical fields, do not act on Shadow Sector \( \bar{g}_{\mu\nu}, C \)
- The \( \chi_\mu \) transformations act on Shadow Sector, do not act on Physical Sector
- “Real diffeomorphisms” diagonal subgroup
Sen’s Action

Fields: $q - 1$ form $P$, SD q-form $Q$, $Q = \ast \eta Q$

$$M(Q)_{\mu_1 \ldots \mu_q} = \frac{1}{q!} M_{\mu_1 \ldots \mu_q}^{\nu_1 \ldots \nu_q} Q_{\nu_1 \ldots \nu_q}$$

$$S = \int \left( \frac{1}{2} dP \wedge \ast \eta dP - 2Q \wedge dP - Q \wedge M(Q) \right)$$

Define: $G \equiv \frac{1}{2} (dP + \ast \eta dP) + Q \quad G = \ast \eta G \quad F \equiv Q + M(Q)$

Field equations imply: $dG = 0, \quad dF = 0$

Choose $M(Q)$ so that: $F = \ast F$
Sen’s Action

Fields: $q - 1$-form $P$, SD $q$-form $Q$, $Q = \star \eta Q$

$$M(Q)_{\mu_1 \ldots \mu_q} = \frac{1}{q!} M^{\nu_1 \ldots \nu_q} Q_{\nu_1 \ldots \nu_q}$$

$$S = \int \left( \frac{1}{2} dP \wedge \star \eta dP - 2Q \wedge dP - Q \wedge M(Q) \right)$$

Define:

$$G \equiv \frac{1}{2} (dP + \star \eta dP) + Q$$

$$G = \star \eta G$$

$$F \equiv Q + M(Q)$$

Field equations imply:

$$dG = 0, \quad dF = 0 \quad \implies \quad G = dC, \quad F = dA$$

Choose $M(Q)$ so that:

$$F = \star F$$
New Action

Fields: $q - 1$ form $P$, SD $q$-form $Q$, \[ Q = * Q \quad \eta \rightarrow \bar{g}, \quad * \eta \rightarrow \bar{*} \]

\[ S = \int \left( \frac{1}{2} dP \wedge *dP - 2Q \wedge dP - Q \wedge M(Q) \right) \]

Define: \[ G \equiv \frac{1}{2}(dP + *dP) + Q \quad G = * G \quad F \equiv Q + M(Q) \]

Field equations imply: \[ dG = 0, \quad dF = 0 \]

Choose $M(Q)$ so that: \[ F = * F \]
New Action

Fields: \( q \)-1form \( P \), SD \( q \)-form \( Q \), \( Q = \ast Q \)  \( \eta \rightarrow \bar{g} \), \( \ast \eta \rightarrow \bar{\ast} \)

\[
S = \int \left( \frac{1}{2} dP \wedge \bar{\ast} dP - 2Q \wedge dP - Q \wedge M(Q) \right)
\]

Define: \( G \equiv \frac{1}{2} (dP + \bar{\ast} dP) + Q \) \( G = \ast G \) \( F \equiv Q + M(Q) \)

Field equations imply: \( dG = 0, \) \( dF = 0 \) \( \implies G = dC, \) \( F = dA \)

Choose \( M(Q) \) so that: \( F = \ast F \)
Dependence on Metrics

Term in action \(-\int Q \wedge M(Q)\) gives interaction between \(Q, g, \bar{g}\)

Action gives complicated theory of \(P, Q, g, \bar{g}\)

But gives simple theory of

\[ G \equiv \frac{1}{2}(dP + \tilde{*}dP) + Q \]

\[ F \equiv Q + M(Q) \]

with \(F\) interacting with \(g\) and \(G\) interacting with \(\bar{g}\), but no interactions between the physical sector \(F, g\) and the shadow sector \(G, \bar{g}\)
Bi-Metric Geometry

Interpolating Structure \( f_\mu^\nu \)

\[ g_{\mu\nu} = f_\mu^\rho f_\nu^\sigma \tilde{g}_{\rho\sigma} \]

Generalisation of vielbein

Map on forms \( \Phi : X \rightarrow \Phi(X) \)

\[ \Phi(X)_{\mu_1...\mu_r} = f_{\mu_1}^{\alpha_1}...f_{\mu_r}^{\alpha_r}X_{\alpha_1...\alpha_r} \]

converts between the two Hodge duals for the two metrics

\[ * \Phi(X) = \Phi(\tilde{*}X) \]

maps \( \tilde{g} \)-self-dual forms to \( g \)-self-dual forms
Bi-Metric Geometry

**Interpolating Structure** \( f_{\mu}^{\nu} \)

\[
g_{\mu \nu} = f_{\mu}^\rho f_{\nu}^\sigma \tilde{g}_{\rho \sigma} \quad \text{Generalisation of vielbein}
\]

**Map on forms** \( \Phi : X \rightarrow \Phi(X) \)

\[
\Phi(X)_{\mu_1 \ldots \mu_r} = f_{\mu_1}^{\alpha_1} \ldots f_{\mu_r}^{\alpha_r} X_{\alpha_1 \ldots \alpha_r}
\]

converts between the two Hodge duals for the two metrics

\[
* \Phi(X) = \Phi(\tilde{*}X)
\]

e.g. \( f_{\mu}^{\nu} = e_{\mu}^{a} \overline{e}_{a}^{\nu} \)

maps \( \tilde{g} \)-self-dual forms to \( g \)-self-dual forms
Bi-Metric Geometry

Interpolating Structure \( f_{\mu}^{\nu} \)

\[ g_{\mu\nu} = f_{\mu}^{\rho} f_{\nu}^{\sigma} \tilde{g}_{\rho\sigma} \]

Generalisation of vielbein

Map on forms \( \Phi : X \to \Phi(X) \)

\[ \Phi(X)_{\mu_1...\mu_r} = f_{\mu_1}^{\alpha_1} ... f_{\mu_r}^{\alpha_r} X_{\alpha_1...\alpha_r} \]

converts between the two Hodge duals for the two metrics

\[ * \Phi(X) = \Phi(\tilde{*}X) \]

maps \( g \)-self-dual forms to \( \tilde{g} \)-self-dual forms

Can be global tensor field, or defined locally in patches
Given $Q = \bar{*}Q$, aim is to construct $F = Q + M(Q)$ such that $F = *F$ implies $\Phi^{-1}(F) = U$ for some $\bar{*}U = U$. 
Given $Q = \bar{*} Q$ aim is to construct

$$F = Q + M(Q)$$

such that

$$F = * F$$

$$F = * F \implies \Phi^{-1}(F) = U$$

for some $\bar{*} U = U$

Then

$$F = Q + M = \Phi(U)$$

Write as linear operator $\Phi U$
Projectors

\[ \tilde{\Pi}_\pm = \frac{1}{2}(1 \pm \bar{\pi}) \quad F_\pm = \tilde{\Pi}_\pm F \]

\[ F_+ = Q \]

\[ F_- = M \]
Projectors

\[ \tilde{\Pi}_\pm = \frac{1}{2} (1 \pm \bar{\Phi}) \]

\[ F_\pm = \tilde{\Pi}_\pm F \]

\[ F = \Phi U \]

\[ F_+ = Q = \tilde{\Pi}_+ \Phi U = NU \]

\[ F_- = M \]

\[ N \equiv \tilde{\Pi}_+ \Phi \tilde{\Pi}_+ \]
Projectors

\[ \Pi_{\pm} = \frac{1}{2} (1 \pm \bar{\pi}^*) \]

\[ F_{\pm} = \Pi_{\pm} F \]

\[ F = \Phi U \]

\[ F_+ = Q = \Pi_+ \Phi U = NU \]

\[ F_- = M = \Pi_- \Phi U = KU \]

\[ N \equiv \Pi_+ \Phi \Pi_+ \]

\[ K \equiv \Pi_- \Phi \Pi_+ \]
Projectors

\[ \bar{\Pi}_\pm = \frac{1}{2}(1 \pm \bar{\Phi}) \]

\[ F_\pm = \bar{\Pi}_\pm F \]

\[ F = \Phi U \]

\[ F_+ = Q = \bar{\Pi}_+ \Phi U = NU \]

\[ F_- = M = \bar{\Pi}_- \Phi U = KU \]

\[ N \equiv \bar{\Pi}_+ \Phi \bar{\Pi}_+ \]

\[ K \equiv \bar{\Pi}_- \Phi \bar{\Pi}_+ \]

\{ Eliminate U \}
Projectors

\[ \Pi_{\pm} = \frac{1}{2} (1 \pm \bar{\bar{\Pi}}^*) \]

\[ F_{\pm} = \Pi_{\pm} F \]

\[ F = \Phi U \]

\[ F_+ = Q = \Pi_+ \Phi U = NU \]
\[ F_- = M = \Pi_- \Phi U = KU \]

\[ N \equiv \Pi_+ \Phi \Pi_+ \]
\[ K \equiv \Pi_- \Phi \Pi_+ \]

Generalised inverse exists

\[ \tilde{N}N = \Pi_+ \]

\[ U = \tilde{N}Q \]
Projectors

$$\tilde{\Pi}_\pm = \frac{1}{2}(1 \pm \bar{\ast})$$

$$F_\pm = \tilde{\Pi}_\pm F$$

$$F = \Phi U$$

$$F_+ = Q = \tilde{\Pi}_+ \Phi U = NU$$

$$F_- = M = \tilde{\Pi}_- \Phi U = KU$$

$$N \equiv \tilde{\Pi}_+ \Phi \tilde{\Pi}_+$$

$$K \equiv \tilde{\Pi}_- \Phi \tilde{\Pi}_+$$

Generalised inverse exists

$$\tilde{N} N = \tilde{\Pi}_+$$

$$U = \tilde{N} Q$$

Then this gives

$$M(Q) = KU = K\tilde{N}Q$$
Projectors\[
\tilde{\Pi}_\pm = \frac{1}{2}(1 \pm \bar{\Phi})
\] \[
F_\pm = \tilde{\Pi}_\pm F
\]
\[
F_+ = Q = \tilde{\Pi}_+ \Phi U = NU
\]
\[
F_- = M = \tilde{\Pi}_- \Phi U = KU
\]
\[
N \equiv \tilde{\Pi}_+ \Phi \tilde{\Pi}_+
\]
\[
K \equiv \tilde{\Pi}_- \Phi \tilde{\Pi}_+
\]

Generalised inverse exists \[
\tilde{N}N = \tilde{\Pi}_+
\]
\[
U = \tilde{N}Q
\]

Then this gives \[
M(Q) = KU = K\tilde{N}Q
\]
\[
M(Q) \text{ globally well-defined, even if interpolating structure only defined locally}
\]
2d Chiral Boson

Zweibein $\bar{e}_\mu^a$ for $\bar{g}$, \( a, b = \pm \), \( \bar{e}^\pm = 2^{-1/2} (\bar{e}^0 \pm \bar{e}^1) \), \( \partial_a = \bar{e}_a^\mu \partial_\mu \)

\[
S = \int d^2 x \sqrt{\bar{g}} \left( \partial_+ P \partial_- P + 2Q_+ \partial_- P + M_- Q_+ Q_+ \right)
\]

\[
G_+ = \frac{1}{2} \partial_+ P + Q_+, \quad F_+ = Q_+, \quad F_- = M_- Q_+
\]

Field equations give

\[
G = \bar{G} \quad F = * F
\]

if M chosen to be:

\[
M_- = \frac{\mathcal{D}}{1 + \frac{1}{2} \mathcal{D} g^{\lambda \tau} \bar{g}_{\lambda \tau}} g^{++}
\]

\[
\mathcal{D} = \frac{1}{2} \left[ (\bar{g}^{\mu \nu} g_{\mu \nu})^2 - \bar{g}^{\mu \nu} g_{\nu \rho} \bar{g}^{\rho \sigma} g_{\sigma \mu} \right]
\]

\[
g^{++} = g^{\mu \nu} \bar{e}_\mu^+ e_\nu^+
\]
Symmetries

- Two metrics: 2 kinds of “diffeomorphism” symmetries

\[ \delta g_{\mu\nu} = 2 \partial_{(\mu} \zeta_{\nu)} + \ldots, \quad \delta \bar{g}_{\mu\nu} = 2 \partial_{(\mu} \chi_{\nu)} + \ldots \]

- Extends to symmetries of full theory

- The \( \zeta_{\mu} \) transformations act on Physical Sector \( g_{\mu\nu}, A + \) other physical fields, do not act on Shadow Sector \( \bar{g}_{\mu\nu}, C \)

- The \( \chi_{\mu} \) transformations act on Shadow Sector, do not act on Physical Sector

- “Real diffeomorphisms” diagonal subgroup
Symmetries

The $\zeta_\mu$ transformations act on Physical Sector $g_{\mu\nu}, A + \text{other physical fields}$, do not act on Shadow Sector $\bar{g}_{\mu\nu}, C$

$$\delta G = 0, \quad \delta \bar{g} = 0, \quad \delta g = \mathcal{L}_\zeta g$$

$$\delta Q = -\frac{1}{2}(1 + \bar{\kappa})d\delta P, \quad \delta P = i_\zeta F$$

This implies

$$\delta F \approx di_\zeta F = \mathcal{L}_\zeta F$$

plus terms which vanish on-shell
The two metrics

- 2 kinds of mass/energy: physical mass and shadow mass
- Treat $g_{\mu\nu}$ as metric tensor field in usual way, giving physical gravitational field
- Conventional: take $\bar{g}_{\mu\nu}$ to be a 2nd metric tensor field, transition functions involve diffeomorphisms $\delta \bar{g}_{\mu\nu} = 2 \partial_{(\mu} \xi_{\nu)} + \ldots$
The two metrics

- 2 kinds of mass/energy: physical mass and shadow mass
- Treat $g_{\mu\nu}$ as metric tensor field in usual way, giving physical gravitational field
- Conventional: take $\tilde{g}_{\mu\nu}$ to be a 2nd metric tensor field, transition functions involve diffeomorphisms $\delta \tilde{g}_{\mu\nu} = 2 \partial_{(\mu} \xi_{\nu)} + \ldots$

\[
\text{Diffeomorphism} \quad \phi : x \to x' = \phi(x), \quad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = [\phi_*g]_{\mu\nu}(x')
\]

\[
\text{Infinitesimal:} \quad x'^\mu = x^\mu - \xi^\mu + \ldots, \quad g'_{\mu\nu}(x) = g_{\mu\nu}(x) + 2 \partial_{(\mu} \xi_{\nu)} + \ldots
\]
The two metrics

- 2 kinds of mass/energy: physical mass and shadow mass
- Treat $g_{\mu\nu}$ as metric tensor field in usual way, giving physical gravitational field
- Conventional: take $\bar{g}_{\mu\nu}$ to be a 2nd metric tensor field, transition functions involve diffeomorphisms $\delta \bar{g}_{\mu\nu} = 2 \partial_{(\mu} \xi_{\nu)} + \ldots$
The metrics

- 2 kinds of mass/energy: physical mass and shadow mass
- Treat $g_{\mu\nu}$ as metric tensor field in usual way, giving physical gravitational field
- Conventional: take $\bar{g}_{\mu\nu}$ to be a 2nd metric tensor field, transition functions involve diffeomorphisms $\delta \bar{g}_{\mu\nu} = 2 \partial_{(\mu} \xi_{\nu)} + \ldots$
- Unconventional: take it to be a gauge field, allow spin-2 gauge transformations in transition functions: $\delta \tilde{g}_{\mu\nu} = 2 \partial_{(\mu} \xi_{\nu)} + 2 \partial_{(\mu} \chi_{\nu)} + \ldots$
**Unconventional case**

- **Unconventional**: take it to be a gauge field, allow spin-2 gauge transformations in transition functions: 
  \[ \delta g_{\mu \nu} = 2 \partial_{(\mu} \xi_{\nu)} + 2 \partial_{(\mu} \chi_{\nu)} + \ldots \]

- **Particular case**: 
  \[ \bar{\xi}_\mu = -\chi_\mu; \quad \delta \bar{g}_{\mu \nu} = 0 ! \]

- e.g. 
  \[ \bar{g}_{\mu \nu} = \eta_{\mu \nu} ! \]
Interactions

Non-linear Born-Infeld-type interactions

\( M(Q) \) non-linear in \( Q \)

Gives

\[ F_\pm = R(F_\mp) \]

\[ R(Q) \sim \frac{\partial M}{\partial Q} \]

Chern-Simons-type interactions

Add matter couplings to action. Gives:

\[ F = dA + \Omega \]

\[ F = * F \]
Coupling to Branes

p-form gauge field $A$ couples to $p-1$ brane current $j$ (a p-form) by

$$\int A \wedge *j$$

2 problems for self-dual fields:

1. $A$ not a field appearing in action
2. Branes for self-dual fields are dyonic, so need to could to both an electric and a magnetic current
Coupling to Branes

F: q form field strength

\[ d^\dagger F = j \quad \text{and} \quad d^\dagger * F = \tilde{j} \]

Electric $q - 1$ form current $j$, magnetic $q - 1$ form current $\tilde{j}$

E.g. sources localised on $q - 2$ brane

\[ d^\dagger = * d * \]
Coupling to Branes

F: q form field strength

\[ d^\dagger F = j \quad d^\dagger * F = \tilde{j} \quad d^\dagger = * d * \]

Electric \( q - 1 \) form current \( j \), magnetic \( q - 1 \) form current \( \tilde{j} \)

E.g. sources localised on \( q - 2 \) brane

If self-dual: \( F = * F \)

then source “self-dual” \( \tilde{j} = j \) q-2 brane “dyonic”
General non-self-dual case

If $\tilde{j} \neq 0$ then $dF \neq 0$ and there is no potential $A$ on brane so no coupling $A \cdot j$ possible.

Action with both electric and magnetic dynamical sources?
General non-self-dual case

If \( \tilde{j} \neq 0 \) then \( dF \neq 0 \) and there is no potential \( A \) on brane
so no coupling \( A \cdot j \) possible.

Action with both electric and magnetic dynamical sources?

Solution: if \( q = 2 \) introduce Dirac string ending on magnetic monopole, or
if \( q > 2 \) introduce Dirac \( q - 1 \) brane with boundary on magnetic \( q - 2 \) brane.

Dirac string/brane current \( \tilde{J} \)

\[
d^\dagger \tilde{J} = \tilde{j}
\]
Then we have:

\[ d^\dagger \tilde{J} = \tilde{J} \quad \quad d^\dagger \ast F = \tilde{J} \]

\[ \Rightarrow \quad d^\dagger \ast F = d^\dagger \tilde{J} \]

\[ \Rightarrow \quad F = \ast \tilde{J} + dA \]
Then we have:

\[ d^\dagger \tilde{J} = \tilde{j} \quad d^\dagger * F = \tilde{j} \]

\[ \implies d^\dagger * F = d^\dagger \tilde{J} \]

\[ \implies F = * \tilde{J} + dA \]

Action:

\[ S[A] = \int \frac{1}{2} F \wedge * F - A \wedge * j \]

Gives field equations

\[ d^\dagger F = j \]

Dirac

Deser, Gomberoff, Henneaux, Teitelboim
Self-dual case

\[ F = * F \quad \quad d^\dagger F = * dF = j \quad \quad \implies \quad \quad F = dA + J \]

Introduce Dirac brane \( d^\dagger J = j \)
Self-dual case

\[ F = * F \quad d^\dagger F = * dF = j \quad \implies \quad F = dA + J \]

Introduce Dirac brane \( d^\dagger J = j \)

Action

\[ S = \int \left( \frac{1}{2} dP \wedge *dP - 2Q \wedge dP - Q \wedge M(Q) + 2Q \wedge \Omega_- - 2\Omega_+ \wedge M(Q) \right) \]

with \( \Omega = -\Pi_- J \)

\[ \Pi_\pm = \frac{1}{2}(1 \pm *) \]

implies \( F \equiv Q + \Omega_+ + M(Q + \Omega_+) + \Pi_+ J \)

satisfies \( F = * F \quad d^\dagger F = * dF = j \)
Conclusion

- Sen’s action for chiral form fields generalised, OK for general spacetimes
- Extra shadow sector $\bar{g}_{\mu\nu}, C$ which decouples from physical fields
- Shadow sector metric $\bar{g}_{\mu\nu}$ can be background or dynamical
- Good for quantum calculations
- Generalises to allow Born-Infeld and Chern-Simons interactions
- Physical form field $A$ isn’t a fundamental field, but constructed from $P, Q, g, \bar{g}$
- Bi-metric geometry, tensor gauge fields