DOUBLED GEOMETRY VERSUS GENERALISED GEOMETRY

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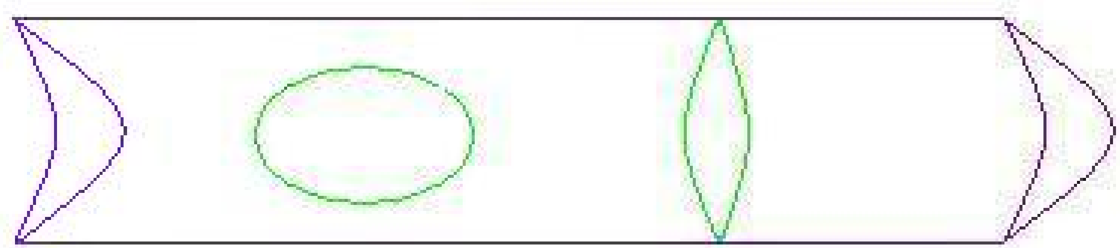
Contents

T-Duality

- One of String theory's most curious features
- A symmetry relating the very large and the very small
- Stems from the ability of strings to wrap compact spaces



- Winding modes give a contribution to a string's energy of the form wR
- Momentum modes give a contribution to a string's energy of the form n/R
- Total energy $wR + n/R$ is unchanged by $R \to 1/R$ and $n \leftrightarrow w$

- More generally, on an n dimensional manifold, T-duality can be described through it's action on a $2n \times 2n$ matix

$$M = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} \tag{1}$$

where g and b are the metric and b-field

- This is

$$M \to hMh^t \,, \tag{2}$$

where $h \in O(n,n;\mathbb{Z})$.

- h can be decomposed into three different types of transformation

1. Theta-shifts

$$h_\Theta = \begin{pmatrix} \mathbb{I}_n & \Theta \\ 0 & \mathbb{I}_n \end{pmatrix} \tag{3}$$

2. Basis changes

$$h_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \tag{4}$$

3. Factorised dualities

$$\begin{pmatrix} \mathbb{I}_n - e_i & e_i \\ e_i & \mathbb{I}_n - e_i \end{pmatrix}, \tag{5}$$

where e_i is a diagonal matrix all of whose entries vanish except for $e_{ii} = 1$. These generalise the $R \to 1/R$ symmetry discussed earlier.

Doubled Geometry

- Doubled geometry renders T-duality geometric by doubling the dimension of a manifold.
- First consider a T^n bundle over a base, N. This becomes a T^{2n} bundle.
- Let Y^m be coordinates in N and $\mathbb{X}^I$ be coordinates in T^{2n}.
- The transition functions of the new manifold are in $O(n,n) \subset GL(2n)$.
- The reduction of the structure group is acheived by introducing the constant metric

$$L = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} . \tag{6}$$

- To construct a string theory, world-sheet one-forms can be introduced

$$P^I = P^I_\alpha \, d\sigma^\alpha = \partial_\alpha \mathbb{X}^I \, d\sigma^\alpha \,. \tag{7}$$

- The string Lagrangian is

$$\mathcal{L}_d = \frac{1}{2} M_{IJ} P^I \wedge * P^J + P^I \wedge J_I + \mathcal{L}(Y) \,, \tag{8}$$

  where M is a positive definite metric on T^{2n}, J_I are source terms depending on Y but not on $\mathbb{X}$, and $\mathcal{L}(Y)$ is the part of the Lagrangian depending on Y only.

- This reproduces the standard string-theory formalism if M is a metric for the coset $O(n,n)/O(n) \times O(n)$.

- A constraint

$$P^I = L^{IJ} * (M_{JK} P^K + J_J) \tag{9}$$

is imposed to reduce the number of dimensions.

- Its consistency requires $G^2 = (LM)^2 = 1$.
- As it stands, this Lagrangian is globally invariant under $GL(2n;\mathbb{Z})$.
- The constraint and presence of L break this to the $O(n,n;\mathbb{Z})$ group of T-duality.
- An element of it, h say, acts as

$$M \to h^t M h\,, \tag{10}$$
$$P \to h^{-1} P\,, \tag{11}$$
$$J \to h^t J \tag{12}$$
$$\mathbb{X} \to h^{-1} \mathbb{X}\,. \tag{13}$$

Chosing a Physical Space

- The physical space is an n-dimensional torus embedded in a 2n-dimensional torus and fibred over some base.
- It can be located by specifying a *polarisation* that splits the fibre coordinates into a physical part X and its dual $\widetilde{X}$.
- With respect to the metric L, both the physical and dual spaces are null, ie. $X_I^i L^{IJ} X_J^j = 0$.
- The splitting can be achieved by introducing two projectors: Π which projects onto the physical space, and $\widetilde{\Pi}$, which projects onto its dual.

- Their action on coordinates is

$$\Phi^I{}_J \mathbb{X}^J = \begin{pmatrix} \Pi^i{}_J \mathbb{X}^J \\ \widetilde{\Pi}^i{}_J \mathbb{X}^J \end{pmatrix} = \begin{pmatrix} X^i \\ \widetilde{X}^i \end{pmatrix} , \tag{14}$$

where I've defined

$$\Phi^I{}_J = \begin{pmatrix} \Pi^i{}_J \\ \widetilde{\Pi}^i{}_J \end{pmatrix} . \tag{15}$$

- Likewise their action on momentum is

$$\Phi^I{}_J P^J = \begin{pmatrix} p^i \\ q^i \end{pmatrix} , \tag{16}$$

- The current J can be similarly decomposed

$$J = \begin{pmatrix} j \\ k \end{pmatrix} . \tag{17}$$

- In this basis, the metric M for the coset space $O(n,n)/O(n) \times O(n)$ can be specified by a symmetric tensor g, and an antisymmetric tensor b, it is given by

$$M = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} . \tag{18}$$

- We can also relate q^i to p^i

$$q = g(*p - k) + bp \, . \tag{19}$$

- The polarisation specifies a *product structure*, a $(1,1)$ tensor obeying

$$\mathcal{R}^2 = 1 \tag{20}$$

- It splits the tangent space in two, each part being an eigenspace of $\mathcal{R}$, one with eigenvalue $+1$, the other with eigenvalue -1.
- On T^{2n} this corresponds to a splitting into two tori T^n when $\mathcal{R} \in GL(n;\mathbb{Z})$ and the periodicity of the coordinates is respected.
- We are interested in cases where these eigenspaces are null with respect to L so

$$L\mathcal{R} + \mathcal{R}^T L = 0\,. \tag{21}$$

T-duality

- The string Lagrangian is invariant under $O(n,n)$ transformations

$$\mathcal{L}_d(h^t M h, h^t J, h^{-1}\mathbb{X}) = \mathcal{L}_d(M, J, \mathbb{X})\,. \tag{22}$$

- However $\mathbb{X} \to h^{-1}\mathbb{X}$ is just a field redefinition, so choices of M and J related by $O(n,n)$ transformations specify the same theory without any corresponding transformation of $\mathbb{X}$.
- To produce a physical theory the polarisation Π also has to be specified.
- Two projectors related by a diffeomorphism of T^n define the same theory. So (M, J, Π) and $(h^t M h, h^t J, d\Pi h)$ are equivalent if $d \in GL(n;\mathbb{Z})$.

- If Π is fixed though, and only M and J changed, the standard form of T-duality results. Alternatively, M and J can be fixed, and the polarisation changed, in which case

$$\mathcal{R} \to h^{-1}\mathcal{R}h\,. \tag{23}$$

- Either the doubled torus can be transformed, with the physical torus fixed, an *active transformation*, or the physical torus can be transformed within the larger, unchanging, doubled torus.
- In the latter case, a *passive transformation*, T-duality amounts to the observation that physics is independent of the choice of physical subspace.

T-Folds

- Remember that doubled geometry begins with a torus T^{2n} fibred over some base N.
- This will have an open cover U_α along with the corresponding transition functions $h_{\alpha\beta}$ relating coordinates on different patches.
- The coordinates of the fibres in different patches are related by

$$\mathbb{X}_\alpha = h_{\alpha\beta}^{-1}\mathbb{X}_\beta + x_{\alpha\beta} \tag{24}$$

on $U_\alpha \cap U_\beta$, where $h_{\alpha\beta}$ is a matrix in $GL(2n, \mathbb{Z})$ and $x_{\alpha\beta}$ is a shift in $U(1)^{2n}$.

- The corresponding transformations for other quantities are

$$P_\alpha = h_{\alpha\beta}^{-1} P_\beta \,, \tag{25}$$

$$L_\alpha = h_{\alpha\beta}^{-1} L_\beta (h_{\alpha\beta}^{-1})^t \,, \tag{26}$$

$$M_\alpha = h_{\alpha\beta}^t M h_{\alpha\beta} \,, \tag{27}$$

$$J_\alpha = h_{\alpha\beta}^t J_\beta \,. \tag{28}$$

- If the metric L is held constant, the transition functions $h_{\alpha\beta}$ are forced to lie in $O(n,n;\mathbb{Z}) \subset GL(2n;\mathbb{Z})$ and these correspond to the rules for T-duality.
- $\mathcal{R}$ has to be constant within any coordinate patch but it can change from one patch to another

- In which case it obeys

$$\mathcal{R}_\alpha = h_{\alpha\beta}^{-1}\mathcal{R}_\beta h_{\alpha\beta}\,. \tag{29}$$

- A T-duality though, occurs when M and J are transformed, but $\mathcal{R}$ is not, or vice versa.
- If the transition functions are all within $GL(n;\mathbb{Z})$, $\mathcal{R}$ would be preserved anyway and the corresponding local patches $U_\alpha \times T^n$ join up to form a spacetime manifold, a T^n bundle over N.
- This doesn't happen generally though, and there are cases where there is no globally defined choice of physical torus within the doubled torus.
- These correspond to non-geometric compactifications of string theory called *T-folds*. Doubled geometry renders them geometric, the transition functions being diffeomorphisms of the larger manifold.

D-Branes

- A *Dp-brane* can be thought of as a p dimensional hypersurface on which open strings are forced to end.
- At the world sheet's boundary

$$\partial_t X^i = 0\,, \tag{30}$$
$$\partial_\sigma X^\mu = 0\,, \tag{31}$$

where $i = 1 ... p$ are directions in the D-brane, and μ are not.

- Dualising a dimension interchanges Dirichlet and Neumann boundary conditions so that if, for instance, X^i is Dirichlet, $\widetilde{X^i}$ is Neumann, and vice-versa.
- In the doubled torus T^{2n} there must be n of each type, it being a combination of an n-torus and its dual.
- If $\Pi_{D,N}$ are the projectors onto Dirichlet and Neumann directions, and $\Pi_N = 1 - \Pi_D^t$

$$\Pi_D \partial_t \mathbb{X}^I = 0\,, \tag{32}$$

$$\Pi_N (M \partial_\sigma \mathbb{X} + J) = 0\,. \tag{33}$$

- Consistency with the constraint

$$\Pi_N L = L \Pi_D \,, \tag{34}$$

which in turn implies

$$\Pi^t_D L \Pi_D = 0 \tag{35}$$

and

$$\Pi^t_N L \Pi_N = 0 \,. \tag{36}$$

- A D-brane can then be regarded as a maximal null subspace which differs from that defining the physical subspace.
- The dimension of the brane p is the dimension of the intersection between this subspace and the physical space.

Doubling Everything

- One curious feature of doubled geometry, as presented thus far, is the fact that only a torus within the compactification manifold is doubled.
- A recent paper by Hull showed how to generalise this to the doubling of the entire manifold.
- In this case, the tangent space of the doubled manifold $\widetilde{M}$ obeys $T\widetilde{M} \simeq (T \oplus T^*)N$ in order to maintain the natural action of $O(n,n)$ upon it, implying that the doubled manifold is the cotangent bundle of M.
- Most of the discussion regarding torus bundles can be carried over to this case.
- It is again possible to introduce a constant $O(n,n)$ invariant metric L and a positive definite metric M obeying $(LM)^2 = 1$.

Generalised Geometry

- In generalised geometry objects that would have been defined on the tangent bundle T, or cotangent bundle T^* are instead defined on their direct sum $T \oplus T^*$.
- There is a natural symmetric bilinear form, the inner product, on $T \oplus T^*$ arising from the pairing of elements in T with those in T^*

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}\left(\xi(Y) + \eta(X)\right) , \tag{37}$$

where $X, Y \in T$ and $\xi, \eta \in T^*$.

- It defines a metric L on $T \oplus T^*$.

- We can write, on an n dimensional manifold and in a basis where indices $I = 1..n$ lie in T and those ranging from $n+1$ to $2n$ lie in T^*,

$$L = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} , \tag{38}$$

implying L has signature (n, n), and so defines the group $O(n, n)$.

- In the same basis we can write the Lie algebra of $SO(n, n)$, $so(T \oplus T^*) = \{\tau | \langle \tau x, y \rangle + \langle x, \tau y \rangle = 0 \; \forall x, y \; \in T \oplus T^*\}$ as

$$\tau = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix} , \tag{39}$$

where $A \in \mathrm{End}(V)$, $B : T \longrightarrow T^*$, $\beta : T^* \longrightarrow T$, and B and β are skew. Then B is a two-form and β is a bivector.

- This demonstrates that $so(T \oplus T^*) = \Lambda^2(T \oplus T^*) = \mathrm{End}(T) \oplus \Lambda^2 T^* \oplus \Lambda^2 T$.
- The action of B

$$\exp(B) = \begin{pmatrix} \mathbb{I}_n & 0 \\ B & \mathbb{I}_n \end{pmatrix}, \tag{40}$$

or $X + \xi \to X + \xi + i_X B$, can be thought of as a shear in the T^* direction.

- Likewise, the action of β

$$\exp(\beta) = \begin{pmatrix} \mathbb{I}_n & \beta \\ 0 & \mathbb{I}_n \end{pmatrix} \tag{41}$$

gives a shear in the T direction.

- Finally, A can be used to embed $GL(n)$ in $SO(n,n)$

$$P \mapsto \begin{pmatrix} P & 0 \\ 0 & {P^*}^{-1} \end{pmatrix}, \tag{42}$$

where $P \in GL(n)$.

Maximal Isotropics

- By combining them, generalised geometry robs T and T^* of their privileged status.
- This can be seen most clearly by considering *maximal isotropic subspaces*, which, roughly speaking, can be thought of as generalisations of the tangent and cotangent bundles.
- A maximal isotropic subspace $L < T \oplus T^*$ is an n-dimensional space on which the inner product always vanishes. That is

$$\langle X + \eta, Y + \xi \rangle = 0 \; \forall \; X + \eta, Y + \xi \in L. \tag{43}$$

- If L and L' are two maximal isotropics that obey $L \cap L' = 0$, $L \oplus L' = T \oplus T^*$ and statements made about T and T^* apply equally to L and L'.
- Both T and T^* are maximal isotropics.

- So is

$$L(E,\eta) = \{X + \xi \in E \oplus V^* \ : \xi|_E = \epsilon(X)\} \tag{44}$$

when ϵ is antisymmetric because then

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X)) = \frac{1}{2}(\epsilon(X,Y)+\epsilon(Y,X)) = 0\,. \tag{45}$$

In fact any maximally isotropic subspace has this form.

- For instance, $L(T,0) = T$ and $L(0,0) = T^*$.
- There is a natural bracket on $T \oplus T^*$ describing flows generated by sections of that bundle in a manner analogous to the Lie bracket, giving the rate of change of, say, $Y + \eta$ under the flow produced by $X + \xi$.

- It is called the *Courant bracket* and has the form

$$[X+\xi, Y+\eta]_c = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)\,, \qquad (46)$$

- The Courant bracket does not, in general, satisfy the Jacobi identity.
- It is useful to define the *Jacobiator* to measure the extent of this failure

$$Jac(A,B,C) = [[A,B]_c,C]_c + [[B,C]_c,A]_c + [[C,A]_c,B]_c = d\mathcal{N}(A,B,C)\,, \qquad (47)$$

where the *Nijenhuis operator*

$$\mathcal{N}(A,B,C) = \frac{1}{3}(\langle [A,B]_c, C\rangle + \langle [B,C]_c, A\rangle + \langle [C,A]_c, B\rangle)\,. \qquad (48)$$

- A maximal isotropic L is involutive with respect to the courant bracket when either

$$\mathcal{N}|_L = 0 \tag{49}$$

or

$$Jac|_L = 0\,, \tag{50}$$

in which case it is integrable and called a *Dirac structure*.

The Metric and b-field

- I have already noted the importance of isotropic subspaces, but it can also be useful to distinguish a positive definite subspace C_+, and its negative definite orthogonal compliment C_-.
- Doing so reduces the structure group on $T \oplus T^*$ from $O(n,n)$ to its maximal compact subgroup $O(n) \times O(n)$, and generates a positive definite metric

$$M = \langle\ ,\ \rangle_{C_+} - \langle\ ,\ \rangle_{C_-}\,. \tag{51}$$

- Using the natural indefinite metric L to identify $T \oplus T^*$ and its dual, we can regard $G = LM$ as a map from $T \oplus T^*$ to itself.

- The ± 1 eigenspace of this map is $C_\pm$, which enables us to find an expression for G in terms of symmetric and antisymmetric tensors; the origin of the metric and b-field in generalised geometry.

$$G = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} . \tag{52}$$

- Even though G can be written

$$G = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{53}$$

and so looks like the B-transform of a Riemannian metric, it is important to note that the b appearing here is not the same as the B appearing in B-transforms, which has to be closed to maintain integrability.

Generalised Product Structures

- Numerous structures can be defined within generalised geometry. Often they are based on objects found within conventional differential geometry.
- For instance, generalisations of complex and Kähler structures have been studied extensively.
- Of most interest to us though, as they most clearly illustrate the relation between doubled and generalised geometries, will be generalised product structures, a generalisation of product structures, which are themselves real analogues of complex structures.
- An *almost product structure* on a manifold M is a map $R : T \to T$ obeying $R^2 = 1$.

- It is possible to project onto its positive and negative eigenspaces using

$$\pi_\pm = \frac{1}{2}(1 \pm R)\,, \tag{54}$$

- This generates a splitting of the tangent bundle

$$T = T^+ \oplus T^-\,, \tag{55}$$

where $\pi_\pm v = v$ for $v \in T^\pm$ and $\pi_\pm v = 0$ otherwise.

- An almost product structure is integrable if T^+ and T^- are involutive with respect to the Lie bracket, in which case it is simply called a *product structure*. Then for any v, w in T

$$\pi_\mp[\pi_\pm v, \pi_\pm w] = 0\,. \tag{56}$$

- A *generalised product structure* then is an endomorphism

$$\mathcal{R} : T \oplus T^* \rightarrow T \oplus T^* \tag{57}$$

obeying

$$\mathcal{R}^2 = 1 \tag{58}$$

and

$$\mathcal{R}^t L = -L\mathcal{R}\,. \tag{59}$$

- The last ensures that its eigenspaces are null with respect to L.

- Again there projectors give the positive and negative eigenspaces of $\mathcal{R}$

$$\Pi_\pm = \frac{1}{2}(1 \pm \mathcal{R})\,. \tag{60}$$

- These define two maximally isotropic subspaces L_+ and L_-, which obey $L_+ \cap L_- = 0$ and

$$\Pi_\pm(X+\eta)_\pm = (X+\eta)_\pm\,, \tag{61}$$
$$\Pi_\pm(X+\eta)_\mp = 0\,, \tag{62}$$

  where $(X+\eta)_\pm \in L_\pm$.

- An integrability condition implies these should be involutive with respect to the Courant bracket.

T-Duality

- Generalised geometry naturally contains the continuous version of the $O(n,n;\mathbb{Z})$ group found in T-duality.
- The matrix M appears when the $O(n,n)$ structure group is broken to $O(n) \times O(n)$ through the choice of positive and negative subspaces in $T \oplus T^*$.
- The action of B-transforms and $GL(n)$ upon it corresponds to the theta shifts and basis transformations occurring in T-duality.
- Factorised dualities though are not generally connected continuously to the identity, and so cannot be found in the Lie algebra of $O(n,n)$.
- There exist several proposal for their inclusion in generalised geometry, but their action on the metric and the bundle $T \oplus T^*$ is broadly similar in all of them.

- On a circle bundle, the action of T-duality upon coordinates can be seen by decomposing the tangent and cotangent bundle, separating parts lying within the circle from those in the base

$$T \oplus T^* = TN \oplus \mathbb{R}X \oplus T^*N \oplus \mathbb{R}\theta \,, \tag{63}$$

where θ is a connection form on the circle, and X a vertical vector field $\theta(X) = 1$. A section of $T \oplus T^*$ can be written $V + uX \oplus \xi + v\theta$, where V is a horizontal vector and ξ the pull back to N. T-duality acts as

$$uX \oplus v\theta \to vX^t \oplus u\theta^t \,, \tag{64}$$

where X^t and θ^t are a vector and one form in the T-dual circle.

- For the metric M the corresponding action is described by the matrix

$$h_\theta = \begin{pmatrix} \mathbb{I}_{n-1} & \mathbf{0}_{n-1,1} & \mathbf{0}_{n-1,n-1} & \mathbf{0}_{n-1,1} \\ \mathbf{0}_{1,n-1} & 0 & \mathbf{0}_{1,n-1} & -1 \\ \mathbf{0}_{n-1,n-1} & \mathbf{0}_{n-1,1} & \mathbb{I}_{n-1} & \mathbf{0}_{n-1,1} \\ \mathbf{0}_{1,n-1} & -1 & \mathbf{0}_{1,n-1} & 0 \end{pmatrix}, \tag{65}$$

whose action on the metric M is

$$M \to M' = h_\theta M h_\theta \,. \tag{66}$$

- Note that both these suggest that, at least as far as the metric and $T \oplus T^*$ are concerned, T-duality can be understood in terms of an interchange of elements of the tangent and cotangent bundles.

D-branes Again

- There is a natural description of D-branes in generalised geometry in which they are related to maximal isotropics.
- Consider an isotropic of the form $L(E,\eta) = \{X + \xi \in E \oplus V^* : \xi|_E = \epsilon(X)\}$.
- This corresponds to a D-brane placed within a manifold in such a way that the tangent bundle of the submanifold it occupies, its world volume, corresponds to E.
- This formulation ensures that all D-branes have the same dimension within the generalised set-up.
- Their dimension on the underlying manifold itself can vary though, being the dimension of E, the isotropic's intersection with the tangent bundle.

A Synthesis?

- A comparison of the preceding two sections makes clear that there are similarities between doubled and generalised geometries; *every* significant object appearing in the discussion about one has an analogue in the other.

1. Metrics Both geometries contain the indefinite metric

$$L = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \tag{67}$$

and the positive definite metric

$$M = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix}, \tag{68}$$

2. Isotropic Subspaces In doubled geometry these define a physical space, in generalised geometry they act as anologues of the tangent and cotangent spaces.
3. Product Structures Doubled geometry relies on a product structure $\mathcal{R}$ to define the physical space. A similar entity, the generalised product structure, can be found in generalised geometry.
4. T-duality The group $O(n,n;\mathbb{Z})$ underlying T-duality can be found in generalised geometry and doubled geometry. In doubled geometry it is the part of the $GL(2n;\mathbb{Z})$ group of diffeomorphisms of the doubled torus that preserves the indefinite metric L, while in generalised geometry $O(n,n)$ is the structure group of the natural metric on $T \oplus T^*$.
5. D-branes D-branes can be incorporated into both doubled and generalised geometry in a similar manner. In both cases they are maximal null subspaces whose physical dimension is that of their intersection with another null subspace.

- Further evidence for this connection is provided by the observation that when an entire manifold is doubled, the tangent bundle of the newly doubled manifold is isomorphic to $T \oplus T^*$ defined on the original.
- This suggests that the similarity between the geometries stems, in part, from the fact that the objects we are interested in, metrics and the like, are defined upon these bundles, and their action on them is the same in both geometries.
- Imagine a theory based on generalised geometry and defined with sections of $T \oplus T^*$.
- This would lead naturally to an indefinite metric L of signature (n, n) from the natural pairing between elements of a vector space and its dual.
- This metric in turn defines an $O(n, n)$ symmetry any such theory would be expected to possess.

- This symmetry might make it impossible to distinguish the tangent bundle from the cotangent bundle.
- A positive definite metric, defined by positive and negative definite subspaces of $T \oplus T^*$, could be generated if it were broken to $O(n) \times O(n)$.
- If, in addition, null subspaces were present, they could define the physical subspace, or act as D-branes.
- The tangent bundle T could always be taken as the physical space, leading to an active picture of T-duality that keeps T fixed while acting on other sections of $T \oplus T^*$, including any defining the metric and b-field.
- Alternatively an action of $O(n, n)$ on the physical space could be allowed, leading to the passive view of T-duality, which would become a symmetry on account of the portability of statements concerning T and T^* to any pair of non-intersecting maximal isotropics.

- Doubling the dimension of a manifold doubles the dimension of its tangent bundle.
- Doubled geometry then imposes additional structures in the form of the metrics L and M, and the product structure $\mathcal{R}$.
- The corresponding objects in generalised geometry have an identical action on $T \oplus T^*$.
- If we were to restrict our interest to the behaviour of these bundles, the two geometries would, to all intents and purposes, be indistinguishable.
- However, a number of differences would still remain due to the differing nature of the manifold on which the bundles are supported.

1. **Transition Functions**

– In doubled geometry the $O(n,n)$ group of T-duality arises as part of the $GL(2n)$ group of transition functions for the coordinates on a $2n$ dimensional manifold, whereas in generalised geometry the transition functions of the underlying manifold's coordinates remain in $GL(n)$.

– This needn't be a problem if we are only interested in the behaviour of bundles, as we would be in the putative theory based on sections of $T \oplus T^*$.

– It might be argued that the presence of a globally defined metric forces their transitions functions to also lie within $GL(n)$, but its existence is not essential in the construction of a generalised geometry, which only requires a manifold with a tangent bundle.

– A metric is not prior to the manifold supporting it.

2. $\mathbf{O(n,n;\mathbb{Z})}$ **Versus** $\mathbf{O(n,n;\mathbb{R})}$

- One of the major differences between doubled and generalised geometries is that the discrete group $O(n,n;\mathbb{Z})$ appears in the former, while the continuous group $O(n,n;\mathbb{R})$ appears in the latter.
- The lattice defining the torus underlying doubled geometry has to be preserved by the transitions functions, forcing them to be discrete, but no such constraint applies in generalised geometry.
- Remember though, that, in the absence of doubled geometry, the discreteness of momentum modes, unlike that of the winding modes, in string theory is a quantum mechanical effect that ensures the wavefunction is single valued.
- It isn't inconceivable that something similar happens in generalised geometry, forcing sections of $T \oplus T^*$ to take discrete values.

– There are indications that something like this might happen to ensure consistency of doubled geometry when the entire manifold is doubled, on compact manifolds, where the cotangent bundle must be quotiented by fiat in order to preserve the boundary conditions, and in the fact that if it didn't the known quantisation of momentum would not be consistent with the $O(n,n)$ symmetry.

3. **Topology Change**

– T-dualities can change the topology, as well as the geometry, of a manifold.

– This might preclude a relationship between doubled and generalised geometry; the topology of differing physical slices may differ in doubled geometry, but it is hard to see how an action on a fibre could alter the topology of its base.

- However, if a theory possessed an $O(n,n)$ symmetry originating in generalised geometry it would not be possible to unambiguously distinguish between the tangent and cotangent spaces, and this might lead to confusion in determining a manifold's topology.

4. **Integrability**
   - In generalised geometry there are integrability conditions that ensure the involutivity of maximal isotropics under the Courant bracket, whose automorphisms are contained within a semi-direct product of diffeomorphisms and B-transforms, rather than the full $O(n,n)$ of T-duality.
   - They need not, however, affect the relationship between doubled and generalised geometry, as the physical subspace is constant on any patch, causing the bracket to vanish.

  - In any case, in active transformations the automatically integrable tangent bundle can be chosen as the physical space, rendering the issue moot.

- The advantage of doubled geometry is that it enables T-duality to be framed in terms of well known geometrical effects, its failing is that it does this at the expense of introducing a new, enlarged manifold. Generalised geometry utilises only objects that would be present on a manifold, but arguably does so in unfamiliar ways.

An Example: The Circle Bundle

- Circle bundles provide the simplest situation in which the relationship between doubled and generalised geometry could be discussed.
- There can be no b field in this case, or B or β transforms; all these involve antisymmetric objects which have to vanish in one dimension.
- The metric M is

$$M = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix} . \tag{69}$$

- In doubled geometry, M is the metric of a torus containing the original circle X, and its dual $\widetilde{X}$.

- Now consider the action of the $O(1,1;\mathbb{Z})$ element

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{70}$$

This can be viewed as a T-duality which either acts, in an active transformation, on M, changing the radius of the physical circle X to $1/R$, or acts passively on the polarisation so that $\widetilde{X}$ becomes physical.

- Imagine patches on the circle bundle $U \times S_R$ and $U' \times S_{R'}$.
- These can be glued together to form a manifold if $R = R'$ on $U \cap U'$.
- In T-folds the circles on different patches are related by the action of h so that $R = 1/R'$, and, as it stands, such a gluing would be impossible.
- The presence of the doubled torus though enables a manifold to be globally defined; X can be glued to $\widetilde{X'}$, both of which have radius R.

- There is no spacetime manifold however as the physical circle has different radii in different patches.
- In generalised geometry the metric is defined by a positive definite definite subspace in which

$$uX = gv\theta = R^2 v\theta \,, \tag{71}$$

taking $x \in T$ and $\theta \in T^*$.

- Under a factorised T-duality this subspace undergoes a transformation

$$v(R^2X + \theta) \to v(X^t + R^2\theta^t)\,, \tag{72}$$

corresponding to a change

$$R \to 1/R\,. \tag{73}$$

- A T-fold could occur when circle bundles over different patches of the base are related by this transformation, an element of $T \oplus T^*$ assigned to T in one patch being assigned to T^* in another.

Conclusions

- This talk has presented evidence of a relationship between generalised geometry and doubled geometry, and, by extension, between it and string theory.
- We have seen that both naturally incorporate T-duality and D-branes, and do so in similar ways.
- I have argued that any theory based upon sections of $T \oplus T^*$ would be expected to reflect these properties, but left the elucidation of its precise details for the future.
- This result relies in part on an assumption that a mechanism exists forcing elements of $T \oplus T^*$ to take discrete values on compact manifolds.

- The objects essential to doubled geometry, an indefinite metric of signature (n, n) that ensures the appearance of the $O(n, n)$ symmetry crucial to T-duality, a positive definite $2n$ dimensional metric combining the b-field with a Riemannian metric, and a product structure that specifies the location of an n dimensional physical space, all have natural analogues in generalised geometry. In the latter case, they act on the bundle $T \oplus T^*$, which is isomorphic to the tangent bundle of the doubled manifold, accounting in part for the similarities between the geometries.
- Doubled geometry is an attempt to describe T-duality within the framework of conventional differential geometry. It does this by accommodating the $O(n, n)$ symmetry of the duality within the diffeomorphism group of a $2n$ dimensional manifold combining the original manifold and its dual. While this successfully provides a geometric account of T-duality, it is unclear if any more fundamental motivation can be provided for the enlargement of the space.

- Conversely, a generalised geometry could be constructed on any manifold. In so far as doubled geometry provides an account of the features of string theory, and in so far as that account can be reproduced in generalised geometry, generalised geometry alone could provide a geometric basis for those features without adding anything new to a manifold.
- It might be argued that the formation of a theory based upon the bundle $T \oplus T^*$ is unnatural, but the structures contained within generalised geometry could be found whenever a manifold has a tangent bundle.
- It is, if anything, their absence from physics that would be surprising.