Quantum Entropy Function and Localization

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Summary

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- Supersymmeteric extremal black holes typically have a near horizon geometry of the form AdS₂ × K, where K is a compact space.
- Quantum entropy function is a proposal which relates the degeneracy d_{hor} of a single centred black hole with the partition function Z_{AdS2} of string theory on AdS₂ × K. This relation takes the form

$$d_{hor} = \left\langle exp\left[-iq_i \oint d\theta A_{\theta}^{(i)}\right] \right\rangle_{AdS_2}^{finite}$$
(1)

where $<>_{AdS_2}$ denotes unnormalized path integral weighted by e^{-A} , A is the euclidean action.

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- This partion function suffers from infrared divergence because of infinite volume of AdS_2 . But there is a unambiguous procedure for extracting its finite part.
- \bullet Consider AdS_2 as a Poincare disk. Then metric has the form

$$ds^2 = \nu (d\eta^2 + \sinh^2 \eta d\theta^2)$$
 (2)

We regularize the infinite volume by putting cut-off $\eta = \eta_0$. Let *L* be the length of the boundary. Then in the limit of large cut-off, the partition function has the form

$$Z_{AdS_2} = e^{CL + O(L^{-1})} \times \Delta \tag{3}$$

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where C and Δ are independent of L. The finite part of the Z_{AdS_2} is defined to be Δ and is called 'quantum entropy function'.

Since the results for microscopic degeneracy are known in most cases, in order to compare with microscopic degeneracy formula one need to take into account multicentred black holes by taking appropriate product of single centred black hole degeneracy.

$$d(\vec{q}) = \sum_{n} \sum_{\{\vec{q}_i\}, \vec{q}_{hair}} \prod_{i=1}^{n} d_{hor}(\vec{q}_i) \ d_{hair}(\vec{q}_{hair}; \{\vec{q}_i\})$$
(4)

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- Consider an arbitrary quantum field theory with function space *M* over which one wish to integrate.
- Let F be a supergroup of symmetries generated by Q and a compact U(1) generator X such that Q² = X.
- Suppose *F* acts freely on *M*. In that case one can form quotient space *M*/*F*. A point in the space *M*/*F* corresponds to an orbit of the elements of *F*. This orbit contains the point and it's images under the action of the supergroup *F*.

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• Thus by integrating first over orbit, one can reduce the integral to an integral over M/F. The integral over orbit is simple and gives a factor of vol(F).

$$\int_{M} e^{-L} \mathbb{O} = vol(F) \int_{M/F} e^{-L} \mathbb{O}$$
 (5)

• Since the integration over the bosonic parameter gives a finite result, the volume of the group *F* is zero.

$$vol(F) = \int dx d\theta = 0$$
 (6)

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- In general the group F does not act freely and has fixed point locus M₀.
- Let C be an arbitrary neighborhood of M_0 and let M' be it's complement. Then the path integral restricted to M' vanishes and the enitre contribution come from the integration over C.
- Since the neighborhood C is arbitrary, the integral in this sense is said to be localised on M_0 .
- The integral is given by the intgration over M_0 weighted by the one loop determinant of the transeverse degree of freedom.

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Motivation:

Our motivation of the work is to understand the quantum entropy function by explicitly doing the path integral.

For this we want a method for systematically calculating the contribution of α' and quantum corrections to entropy of extremal black.

Since the degeneracy formula for BPS black hole is known, this would give a consistency check of this formula.

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Results:

The global symmetry group of $AdS_2 \times S^2$ is SU(1,1|2). It is possible to construct supergroup H_1 which is an analogue of F. This supergroup is generated by supercharge Q_1 and compact bosonic generator $(\hat{L}_0 - J^3)$.

$$Q_1^2 = 4(\hat{L}_0 - J^3), \quad [Q_1, \hat{L}_0 - J^3] = 0$$
 (7)

- ② Using the arguements of localization, we will show that the path integral can receives non-vanishing contribution only from integration around H_1 invariant field configurations.
- We will show the explicit construction of a class of H₁ invariant saddle points.

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- In four space time dimension supersymmetry requires the black holes to be spherically symmetric and hence the near horizon geometry has an $AdS_2 \times S^2$ factor.
- In known examples like $\frac{1}{4}$ BPS black holes in N = 4 supersymmetric theories, $\frac{1}{2}$ BPS black holes in N = 2 supersymmetric theories, the near horizon isometry group $SL(2,\mathbb{R}) \times SO(3)$ gets enhanced to the SU(1,1|2) supergroup.

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The generators of the supergroup SU(1,1|2) satisfy the following algebra

$$[L_{m}, L_{n}] = i(m-n)L_{m+n},$$

$$[J^{3}, J^{\pm}] = \pm J^{\pm}, \qquad [J^{+}, J^{-}] = 2J^{3}$$

$$[L_{n}, G_{r}^{\alpha\pm}] = i(\frac{n}{2} - r)G_{r+n}^{\alpha\pm}$$

$$[J^{3}, G_{r}^{\alpha\pm}] = \pm \frac{1}{2}G_{r}^{\alpha\pm}, \qquad [J^{\pm}, G_{r}^{\alpha\mp}] = G_{r}^{\alpha\pm} \qquad (8)$$

$$\{G_{r}^{+\alpha}, G_{s}^{-\beta}\} = 2\epsilon^{\alpha\beta}L_{r+s} - 2i(r-s)(\epsilon\sigma^{i})_{\beta\alpha}J^{i},$$

$$\epsilon^{+-} = -\epsilon^{-+} = 1, \quad \epsilon^{++} = \epsilon^{--} = 0, \quad m, n = 0, \pm 1, \quad r, s = \pm \frac{1}{2},$$

$$\alpha, \beta = \pm$$

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Here L_n denote the virasoro generator acting on the upper half plane labelled by the coordinate u and is given as

$$L_n = -iu^{n+1}\partial_u - i\bar{u}^{n+1}\partial_{\bar{u}} \tag{9}$$

With this generator, the elements of group $SL(2,\mathbb{R})$ is of the form $exp(is_nL_n)$ with real s_n .

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However we will work in a coordinate w which represent AdS₂ as a disk and this is related to upper half plane coordinate u via

$$w = \frac{1 + iu}{1 - iu} \tag{10}$$

The metric of AdS_2 is

$$ds^2 = 4v \frac{dwd\bar{w}}{(1 - w\bar{w})^2} \tag{11}$$

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In this coordinate the virasoro generator takes the form

$$L_n = \frac{i}{2} [i^n (1+w)^{1-n} (1-w)^{1+n} \partial_w + c.c.]$$
(12)

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We now define

$$\hat{L}_{0} = \frac{1}{2}(L_{1} + L_{-1}), \quad \hat{L}_{\pm} = L_{0} \pm \frac{i}{2}(L_{1} - L_{-1}), \quad \hat{G}_{\pm}^{\alpha\beta} = G_{1/2}^{\alpha\beta} \mp i G_{-1/2}^{\alpha\beta}$$
(13)

In the disk coordinate

$$\hat{L}_{0} = (w\partial_{w} - \bar{w}\partial_{\bar{w}}), \quad \hat{L}_{+} = -i(w^{2}\partial_{w} - \partial_{\bar{w}}), \quad \hat{L}_{-} = i(\partial_{w} - \bar{w}^{2}\partial_{\bar{w}})$$
(14)

In this coordinate \hat{L}_0 has a interpretation of generator of rotation about the origin in the *w*-plane.

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An element of the form

$$exp[i(\zeta^{0}\hat{L}_{0}+\zeta^{+}\hat{L}_{+}+\zeta^{-}\hat{L}_{-}+\eta_{3}J^{3}+\eta_{+}J^{+}+\eta_{-}J^{-})]$$
(15)

will be an element of $SL(2,\mathbb{R}) imes SU(2)$ if

$$(\zeta^{0})^{*} = \zeta^{0}, \quad (\zeta^{\pm})^{*} = \zeta^{\mp}, \quad (\eta_{3})^{*} = \eta_{3} \quad , (\eta_{\pm})^{*} = \eta_{\mp}$$
 (16)

Similarly we get the following reality conditon on the grassman parameters

$$(\theta_{\alpha\beta}^{\gamma})^* = \epsilon^{\alpha\alpha'} \epsilon^{\beta\beta'} \theta_{\alpha'\beta'}^{-\gamma}$$
(17)

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This conditon is achieved by requiring that if $exp(iT_1)$ and $exp(iT_2)$ are elements of SU(1,1|2) then $exp([T_1, T_2])$ must also be an element of the group.

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We define

$$Q_{1} = \hat{G}_{+}^{++} + \hat{G}_{-}^{--}, \quad Q_{2} = -i(\hat{G}_{+}^{++} - \hat{G}_{-}^{--})$$
(18)
$$Q_{3} = -i(\hat{G}_{+}^{-+} + \hat{G}_{-}^{+-}), \quad Q_{4} = \hat{G}_{+}^{-+} - \hat{G}_{-}^{+-}$$

In that case we have

$$\{Q_i, Q_j\} = 8\delta_{ij}(\hat{L}_0 - J^3),$$
(19)

$$[\hat{L}_0 - J^3, Q_i] = 0, \qquad (20)$$

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Subgroup of SU(1,1|2) generated by above subalgebra is denoted as H_0 .

Another subgroup of SU(1,1|2) which will be relevant is H_1 generated by Q_1 and $(\hat{L}_0 - J^3)$.

Also the general element $g \in SU(1,1|2)$ is

$$g = exp[i\{\bar{\zeta}\hat{L}_{+} + \zeta\hat{L}_{-} + \bar{\eta}J^{+} + \eta J^{-} + \theta_{\alpha+}\hat{G}_{-}^{\alpha+} + \theta_{\alpha-}\hat{G}_{+}^{\alpha-}\}]$$

$$\times exp[i\sigma(\hat{L}_{0} + J^{3})] \times exp[i\{\sum_{k=1}^{4}\chi_{k}Q_{k} + \tilde{\sigma}(\hat{L}_{0} - J^{3})\}]$$
(21)

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- In order to carry out the path integral over infinite no. of modes, we will first fix the order of integration.
- We will adopt the following definition of path integral:
 - First we will integrate over the orbits of the subgroup H_1 generated by Q_1 and $(\hat{L}_0 J^3)$,
 - 2 then carry out the integration over the remaining variables in some order.
- We will implicitly assume that the theory admits a formalism in which the H_1 subalgebra generated by Q_1 and $(\hat{L}_0 J^3)$ is realised offshell.

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The division of path integral into the orbits of H_1 and direction transeverse to these can be done by using Fadeev-Popov method. We express the elements of the subgroup H_1 by

$$h = \exp(i\alpha Q_1 + i\beta(\hat{L}_0 - J^3))$$
(22)

Then the path integral can be expressed as

$$\left[\int dh\right] \left[\int e^{-A} \left(\prod_{a} \delta(F^{a})\right) sdet \frac{\delta F^{a}_{\vec{\tau}}}{\partial \tau^{b}}\Big|_{\vec{\tau}=0}\right]$$
(23)

where F^a are a pair of "gauge fixing functionals", $\vec{\tau}$ collectively denotes the transformation parameter (α, β) and $F^a_{\vec{\tau}}$ is the transform of F^a by parameter $\vec{\tau}$.

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- The integration over H₁ has a compact bosonic direction β corresponding to U(1) and a fermionic direction α. Hence the whole integral vanishes.
- If the field configuration ϕ is invariant under only $(\hat{L}_0 J^3)$ then the matrix $\frac{\delta F_{\vec{\tau}}^2}{\partial \tau^b}$ has zero eigen value along the bosonic direction. This makes the superdeterminant vanishes.
- Thus the configuration ϕ must be invariant under both Q_1 and $(\hat{L}_0 J^3)$.

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We choose the coordinates of the field configuration measuring the fluctuations about the field ϕ as follows.

- By Fourier decomposing these fluctuation in (θ φ) coordinate, we can choose them to be eigen functions of (Â₀ J³) with eigen value m. e.g. For scalar field a deformation of the form e^{im(θ-φ)/2}f(θ + φ, r, ψ) will have this property.
- 2 we parametrize all such bosonic fluctuation by z_m^s for m positive and z_m^{s*} for m negative. Here s runs for different value and z_m^s form a complete set bosonic fluctuations with eigen value m.
- Since $Q_1^2 = 4(\hat{L}_0 J^3)$, the action of Q_1 on z_m^s for $m \neq 0$ can not vanish. This action will generate particular fermionic deformation with eigen value m.

$$Q_1 z_m^s = \zeta_m^s \qquad (\hat{L}_0 - J^3) \zeta_m^s = m \zeta_m^s \qquad (24)$$
$$Q_1 \zeta_m^s = 4m z_m^s \qquad (25)$$

Similarly for the complex conjugate deformations, the following relations hold

$$Q_1 z_m^{s*} = \zeta_m^{s*}, \qquad Q_1 \zeta_m^{s*} = -4m z_m^{s*}, \quad m > 0$$
 (26)

 ζ_m^s and ζ_m^{s*} parametrize the complete set of fermionic deformations with $(\hat{L}_0 - J^3)$ eigen values m and -m respectively. We shall call the m = 0 bosonic and fermionic modes collectively as \vec{y}

Now the path integral, around the field ϕ , over the various fields can be regarded as integral over the parameters $z_m^s, z_m^{s*}, \zeta_m^s, \zeta_m^{s*}$ for $m \neq 0$ together with integration over \vec{y} .

$$I = \int d\vec{y} \prod_{m>0,s} dz_m^s dz_m^{s*} d\zeta_m^s d\zeta_m^{s*} \mathscr{I} e^{-A}$$
(27)

Where \mathscr{I} represents any measure factor which might arise from changing the integration variables to $(\vec{y}, \vec{z}, \vec{z^*}, \vec{\zeta}, \vec{\zeta^*})$.

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Now we define

$$I(t) = \int d\vec{y} \prod_{m>0,s} dz_m^s dz_m^{s*} d\zeta_m^s d\zeta_m^{s*} \mathscr{I}e^{-A-tQ_1F}$$
(28)

where t is a real positive parameter and A is the euclidean action satisfying

$$Q_1 A = 0 \tag{29}$$

Here F is

$$F = \sum_{m>0} \sum_{s} z_m^{s*} z_m^s \tag{30}$$

This gives

$$Q_1 F = \sum_{m>0} \sum_{s} \left[4m z_m^{s*} z_m^s + \zeta_m^s \zeta_m^{s*} \right]$$
(31)

Also

$$\partial_t I(t) = \int d\vec{y} \prod_{m>0,s} dz_m^s dz_m^{s*} d\zeta_m^s d\zeta_m^{s*} \mathscr{I}(-Q_1 F) e^{-A - tQ_1 F} \quad (33)$$

$$\partial_t I(t) = -\int d\vec{y} \prod_{m>0,s} dz_m^s dz_m^{s*} d\zeta_m^s d\zeta_m^{s*} \mathscr{I}Q_1(Fe^{-A-tQ_1F}) = 0$$
(34)

Where we have used the Q_1 invariance of the measure. Thus I(t) is independent of t and has the same value in the limits $t \rightarrow 0+$ and $t \rightarrow \infty$. Thus

$$I = \lim_{t \to \infty} \int d\vec{y} \prod_{m > 0, s} dz_m^s dz_m^{s*} d\zeta_m^s d\zeta_m^{s*} \mathscr{I} e^{-A - t \sum_{m > 0} \sum_s [4m z_m^s z_m^{s*} + \zeta_m^s \zeta_m^{s*}]}$$
(35)

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Thus up to an overall t independent normalization constants, the $e^{-t\Sigma_{m>0}\Sigma_s[4mz_m^s z_m^{s*} + \zeta_m^s \zeta_m^{s*}]}$ term on the $t \to \infty$ limit is equivalent to inserting in the path integral a factor

$$\prod_{m>0} \prod_{s} \delta(z_m^s) \delta(z_m^{s*}) \delta(\zeta_m^s) \delta(\zeta_m^{s*})$$
(36)

This shows that the path integral is localized in the subspace of $(\hat{L}_0 - J^3)$ invariant deformations parametrised by the coordinates \vec{y} and the inegral becomes

$$I = \int d\vec{y} \mathscr{I}'(\vec{y}) e^{-A(\vec{y})}$$
(37)

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We can further localize the \vec{y} integral onto Q_1 invariant subspace. Let $(\vec{w}^{\alpha}, \zeta^a)$ are the bosonic and fermionic componets of \vec{y} . Then

$$Q_1 \zeta^a = f^a(\vec{w}^\alpha, \vec{\zeta}) \tag{38}$$

for some functions f^a . We now insert into the path integral a term

$$exp[-tQ_1\sum_{a}\zeta^a f^a(\vec{w},\vec{\zeta})] = exp[-t\sum_{a}f^a(\vec{w},\vec{\zeta})f^a(\vec{w},\vec{\zeta})] \quad (39)$$

Further using nilpotency of Q_1 and Q_1 invariance of the original action, it can be shown that the path integral is independent of t. In the $t \to \infty$, the integral is localised in the bosonic sector onto Q_1 invariant subspace $f^a(\vec{w}, 0) = 0$.

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Result: We have established that nonvanishing contribution to path integral can come from integration around the H_1 invariant field ϕ . The integral is given as integration over H_1 invariant slice passing through the field configuration ϕ and is given by

$$I = \int d\vec{y'} \tilde{\mathscr{I}'}(\vec{y'}) e^{-A(\vec{y'})}$$
(40)

where

$$Q_1 \vec{y}' = 0, \qquad (\hat{L}_0 - J^3) \vec{y}' = 0$$
 (41)

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We consider type *II*B string theory on K3 and consider six dimensional geometries which are asymptotic to $S^1 \times \tilde{S}^1 \times AdS_2 \times S^2$ with background 3-form fluxes. The simplest H_1 -invariant saddle point is

$$ds^{2} = v(d\eta^{2} + \sinh^{2}\eta d\theta^{2}) + u(d\psi^{2} + \sin^{2}\psi d\phi^{2}) + \frac{R}{\tau_{2}}|dx^{4} + \tau dx^{5}|^{2}$$

$$G' = \frac{1}{8\pi^2} [Q_I \sin \psi dx^5 \wedge d\psi \wedge d\phi + P_I \sin \psi dx^4 \wedge d\psi \wedge d\phi + dual]$$
$$V_I^i = constant, \qquad V_I^r = constant \qquad (42)$$
$$1 \le I \le 26, \quad 1 \le i \le 5, \quad 6 \le r \le 26$$

This background is invariant under full SU(1,1|2) symmetry group. By taking orbifold of this background by discrete group \mathbb{Z}_s , one can construct other H_1 invariant saddle point.

- Since the subgroup H₁ contains both Q₁ and Q₁², the generator of Z_s must commute with Q₁.
- The only bosonic generator which commute with Q_1 is $(\hat{L}_0 J^3)$.
- The orbifold of the background is given by the \mathbb{Z}_{s} transformation

$$(heta, \phi, x^5) \rightarrow (heta + rac{2\pi}{s}, \phi - rac{2\pi}{s}, x^5 + rac{2\pi k}{s})$$
 (43)
 $k, s \in \mathbb{Z}$ $gcd(s, k) = 1$

Since $(\hat{L}_0 - J^3)$ shifts θ and ϕ in the opposite direction, the above transformation is generated by $(\hat{L}_0 - J^3)$ together with shifts along x^5 .

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After taking the orbifold

$$ds^{2} = v(d\eta^{2} + \sinh^{2}\eta \left(1 + \frac{(1 - s^{-2})e^{-\eta}}{2\sinh\eta}\right)^{2} d\theta^{2}) + u(d\psi^{2} + \sin^{2}\psi(d\phi + d\theta - s^{-1}d\theta)^{2}) + \frac{R}{\tau_{2}}|dx^{4} + \tau(dx^{5} + ks^{-1}d\theta)|^{2}$$

$$G^{I} = \frac{1}{8\pi^{2}} [Q_{I} \sin \psi (dx^{5} + ks^{-1}d\theta) \wedge d\psi \wedge (d\phi + d\theta - s^{-1}d\theta) + P_{I} \sin \psi dx^{4} \wedge d\psi \wedge (d\phi + d\theta - s^{-1}d\theta) + duaI]$$

$$(44)$$

$$(\theta, \phi, x^5) \equiv (\theta + 2\pi, \phi, x^5) \equiv (\theta, \phi + 2\pi, x^5) \equiv (\theta, \phi, x^5 + 2\pi)$$

For large η , this has the same asymptotic behaviour as the $S^1 \times \tilde{S}^1 \times AdS_2 \times S^2$ background.

- Quantum entropy function gives a prescription for computing the degeneracy of the black hole interms of path integral over string fields on the near horizon geometry of extremal black hole.
- Using the global symmetry group SU(1,1|2), we have shown that there exist a subgroup H_1 generated by fermionic generator Q_1 and bosonic compact generator $(\hat{L}_0 J^3)$ such that $Q_1^2 = 4(\hat{L}_0 J^3)$.
- Using the subgroup H_1 , we have shown the path integral can receives nonvanishing contribution only from integration around field configuration ϕ which are invariant atleast under Q_1 and $(\hat{L}_0 J^3)$.

- We have shown that the path integral about each H₁-invariant φ can be expressed as integral over H₁-invariant slice passing through φ.
- We have also shown the construction of a class *H*₁-invariant saddle points from freely orbifolds of near horizon geometry of the black hole.

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