Extremal and non-extremal black strings in supergravity

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Abstract

We present notes to accompany and complement the seminar talk "Extremal and non-extremal black strings in supergravity" presented by the author to the University of Liverpool String Phenomenology division on Tuesday 16th October 2012.

1 Introduction

Where we try to motivate why we're interested in these theories.

I often find it tricky to motivate any sort of work. "It's interesting" and "Nobody's done it" are usually motivation enough for the researcher, but don't tend to convince the wider community to take any interest in the project. One thing that always seems to do the trick, though, is "string theory". Many solitonic solutions in lower-dimensional supergravity theories have their origin in configurations of solitonic objects in string or M-theory [1], which are then compactified over various Calabi-Yau or other manifolds. Understanding such lower-dimensional objects has helped us to understand many of the non-perturbative aspects, dualities, etc. in string and M-theory.

2 Setup

Where we describe the theories.

No matter what theory we're interested in, there are a few basics that we need to get down before we can start looking for 'solutions'. In this talk we'll be dealing with a class of $\mathcal{N} = 2$, $d = (1, 4)$ supergravity theories coupled to an arbitrary number of vector multiplets. I'm not going to assume an in-depth knowledge of supergravity though. In fact, I'm not really going to assume *any* knowledge of supergravity. All we really care about is that we have a theory, which is defined by its field content and by the particular way that field content comes together in an action.

2.1 The field content

Supergravity multiplet. VMs. HMs.

Let's start with looking at the (on-shell) field content of our theory. It's a gravity theory, so we want a spin-2 graviton in there somewhere. For those who remember their $\mathcal{N}=2$ massless representations, you'll recall that, in $\mathcal{N}=2$ theories, the graviton appears in the so-called "supergravity multiplet" along with a pair of spin-$\frac{3}{2}$ gravitini, and a gauge field, the "graviphoton"

$$\text{SUGRA multiplet} = \{\hat{g}_{MN}, \psi_{M\alpha}, \mathcal{A}_M\}.$$

I should say at this point that we'll use upper-case Roman indices from the middle of the alphabet $(M, N, \ldots)$ to denote the 5d spacetime indices, lower-case Roman indices from the middle of the alphabet $(m, n, \ldots)$ to denote the 3d space(time[1]) indices, and those from the middle of the greek alphabet $(\mu, \nu, \ldots)$ on the odd occasion when we need 4-dimensional indices.

Let us note in passing that the bosonic field content for $\mathcal{N}=2$ supergravity is precisely that which you would expect for the familiar Einstein-Maxwell theories (a graviton and a gauge boson). This fact is nicely exemplified in the Reissner-Nordström familiar from your black holes course. Here, the condition that we have an event horizon translates to the relation $M \geq |Q|$ between the mass M and charge Q of the black hole. This is precisely the BPS bound from supersymmetric theories, and tells us, in particular, that extremal Reissner-Nordström black holes are BPS objects. This allows for the construction of the multi-centred Majumdar-Papapetrou solutions.

Okay, back to the field content. The next ingredient for our theory is the $\mathcal{N}=2$ vector multiplet (VM). Again, think back to when you studied massless SUSY representations, and you'll recall that, in $4+1$ dimensions, an $\mathcal{N}=2$ VM contains a gauge boson, a pair of (symplectic Majorana) fermions, and a single real scalar

$$\text{Vector multiplet} = \{\mathcal{A}_M, \lambda^i_\alpha, \phi\}.$$

For the theory we're interested in, we want to take some number n_V of these VMs, which we label by the index $x = 1, \ldots, n_V$.

Before moving on to consider the possible actions we can construct from such a field content, let us mention one more $\mathcal{N}=2$ multiplet: the hypermultiplet (HM). This consists of 4 real scalars and 2 fermions[2]

$$\text{Hypermultiplet} = \{\phi^a, \chi_\alpha\}.$$

[1] We'll actually just be interested in the case where the 3-dimensional theory is Euclidean, but we'll keep things general for the most part.

[2] The nature of the fermions (Weyl, symplectic, Majorana, etc.) depends on the dimension we're interested in. The hypermultiplet takes the same form in all $d \leq 6$ though.

2.2 Special geometry and the 5d action

The 5d action. Special real manifolds. Prepotential. Couplings in the action all depend on prepotential.

It should be clear from our experience with constructing Lagrangians for global SUSY that we can't just mash all our field content together in a Lorentz-invariant way and hope for the best. The fact that we have to impose that our action be SUSY-invariant (here, of course, we impose local SUSY) greatly restricts the type of actions we can write down.

I'm not going to go about constructing the action for $\mathcal{N} = 2$, $d = (1, 4)$ supergravity coupled to n_V vector multiplets for you. For that, you can look in [2]. I'll just give you the result and say a little about it. So, the bosonic part[3] of the theory has the action

$$
\begin{aligned}
S_5 \quad = \quad & \int d^5x \left[\sqrt{\hat{g}} \left(\frac{\hat{R}}{2} - \frac{3}{4} a_{ij}(h) \partial_M h^i \partial^M h^j - \frac{1}{4} a_{ij}(h) \mathcal{F}^i_{MN} \mathcal{F}^{j|MN} \right) \right. \\
& \left. + \frac{1}{6\sqrt{6}} c_{ijk} \epsilon^{MNPQR} \mathcal{F}^i_{MN} \mathcal{F}^j_{PQ} \mathcal{A}^k_R \right] . \qquad (1)
\end{aligned}
$$

What have we got here? Let's first say something about the notation. First, we've combined the graviphoton $\mathcal{A}_M$ from the supergravity multiplet with the n_V gauge bosons $\mathcal{A}^x_M$ from the VMs into the quantities $\mathcal{A}^i_M$ indexed by $i = 1, \ldots, n_V + 1$. The $\mathcal{F}^i_{MN}$ are the corresponding field strengths, as usual. Recalling the fact that each VM came with a real scalar field, it may seem that we now have 1 too many scalar fields h^i in our action. However, they are not independent, but in fact satisfy the constraint

$$\hat{\mathcal{V}}(h) := c_{ijk} h^i h^j h^k = 1,$$

where the function $\hat{\mathcal{V}}(h)$ is called the prepotential. This leaves us with the required number of independent scalar fields. Next, the coupling matrices $a_{ij}(h)$ which appear in front of the kinetic terms for the gauge fields and scalars, and which depend on the values of the scalars, are related to the prepotential via.

$$a_{ij}(h) = -\frac{1}{3} \partial_{h^i} \partial_{h^j} \log \hat{\mathcal{V}}(h). \qquad (2)$$

Finally, we note that the Chern-Simons term also contains a factor, this time proportional to the third derivative of the prepotential.

So we note that all of the couplings in our action depend on this one function, the prepotential $\hat{\mathcal{V}}(h)$. This general phenomenon is familiar from when we build SUSY Lagrangians, where everything depends on a single superpotential.

[3] We're only going to be interested in the bosonic part of supergravity here. For recent work on dimensional reduction of the fermionic sector [3]

3 Dimensional reduction

Where we start calculating.

Now we've written down an action for our theory, we can start writing down some solutions! Right? Well, try it. Take the action (1), work out the Einstein equations, and the equations of motion for the gauge fields and scalars, and try to find some field configurations which satisfy them. To say it's not easy is an understatement: I bet you didn't even *try*! So, let's pause for a moment and think about what sort of solutions we're interested in. By deciding this, we should be able to write down some initial ansatz for our field configuration that satisfies the symmetries, etc. that we want our putative solution to have.

3.1 Isometries

Killing vectors. Fields independent of some directions.

The first thing we might ask for is that the spacetime solution has some isometries. Formally, this means that the metric satisfies

$$\pounds_\xi \hat{g}_{MN} = 0,$$

for some 'Killing vector' ξ, where $\pounds_X$ refers to the Lie derivative along X[4]. In the case $\xi = \partial_z$ for some coordinate z, the Killing equation reduces to the statement that the metric is independent of z. Of course, there's nothing special about the metric: it's just another dynamical field in our theory. What we're really interested in for dimensional reduction is that *all* of our fields are independent of this coordinate z, i.e.

$$\partial_z \Phi(x) = 0,$$

for all fields $\Phi(x)$ in our theory.

In what we've said so far, there's no particular difference between spacelike and timelike isometries. Recall that, a Killing vector $\xi = \xi^M \partial_M$ is called timelike if $\xi^M \xi_M < 0$ and spacelike if $\xi^M \xi_M > 0$. We're going to be considering both in our applications. Timelike isometries should be familiar to you already, for example when seeking stationary solutions. Spacelike isometries most often arise in the context of spherical symmetry, but here we're actually going to impose that our spacetime is translation-invariant along some spacelike direction. This is what makes our solutions "string-like".

Let's briefly summarise where we stand at this moment. We have a 5-dimensional action (1) describing the dynamics of a bunch of fields: a graviton, $n_V + 1$ gauge fields, and $n_V + 1$ constrained scalar fields. Based on the type of symmetries we want our solutions to have, we've made an ansatz that each of our fields is independent of two of our five spacetime directions, call them x^0 and x^4, which (for now) we take to be either spacelike or timelike (up to a maximum of 1 timelike direction).

[4]Yes, I still use a $\pounds$ sign to denote the Lie derivative, blame Schutz's geometry book.

3.2 Kaluza-Klein reduction

KK ansatz for the fields. Example of decomposing matrices, vectors. Dualising p-form gauge fields.

We're now all set to describe the dimensional reduction procedure. We follow the well-known Kaluza-Klein procedure, where we take one or more of the directions in our spacetime to be compact. A Fourier expansion of the fields will then give massless zero modes plus an infinite tower of massive states, with mass depending on the radius of the compactified direction. Dimensional reduction corresponds to throwing away these massive modes and focusing solely on the massless spectrum. For an excellent, readable, review of this technique, see Section 7.1 of [4].

One way of thinking about dimensional reduction is to ask the question: How do D-dimensional fields look to someone living in $d < D$ dimensions? In terms of the examples we're dealing with, we want to know how the various representations of $SO(1,4)$ (the graviton, vectors, and scalars) decompose in terms of representations of $SO(1,2)$ (for Lorentzian signature) or $SO(3)$ (for Euclidean signature).

Let's take things one step at a time. Let x^0 and x^4 be our compactified directions, which can be either timelike or spacelike. Consider the case where we first "reduce" over the x^0 direction. We can consider how each of our $SO(1,4)$ representations look in the 4-dimensional theory.

Clearly, if Φ is a singlet (scalar field) of $SO(1,4)$, it will also be a singlet of $SO(1,3)$ or $SO(4)$ (and, indeed, of the 3-dimensional versions thereof). So all of our 5-dimensional scalars are 4-dimensional scalars.

If Φ_M transforms as a vector under $SO(1,4)$, then we can decompose it as

$$\Phi_M = (\Phi_0, \Phi_\mu),$$

where Φ_0 is a scalar in 4 dimensions and Φ_μ a vector. Likewise, if we want to reduce again over x^4, we can decompose Φ_μ as

$$\Phi_\mu = (\Phi_m, \Phi_4).$$

Hence, we see that an $SO(1,4)$ vector decomposes into a vector and 2 scalars of $SO(1,2)$ or $SO(3)$.

If Φ_{MN} transforms as a rank 2 symmetric tensor under $SO(1,4)$, then we can decompose it as

$$\Phi_{MN} = \begin{pmatrix} \Phi_{00} & \Phi_{0\mu} \\ \Phi_{\mu 0} & \Phi_{\mu\nu} \end{pmatrix},$$

which corresponds to a scalar (Φ_{00}), a vector ($\Phi_{0\mu}$), and a rank 2 symmetric tensor ($\Phi_{\mu\nu}$) of $SO(1,3)$ or $SO(4)$. If we perform a further reduction over x^4, then we can decompose the vector representation as $\Phi_{0\mu} = (\Phi_{04}, \Phi_{0m})$ as before. The rank 2 symmetric representation further decomposes as

$$\Phi_{\mu\nu} = \begin{pmatrix} \Phi_{44} & \Phi_{4m} \\ \Phi_{m4} & \Phi_{mn} \end{pmatrix}.$$

Hence, we see that the rank 2 symmetric representation of $SO(1,4)$ decomposes into 3 scalars, 2 vectors, and a rank 2 symmetric representation of $SO(1,2)$ or $SO(3)$.

Actually, there is a particularly nice way of writing the metric (the rank 2 symmetric tensor) which we will use in calculations: this is known as the Kaluza-Klein ansatz, and is written as

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma}\left(dx^0 + \mathcal{A}^0\right)^2 - \epsilon_2 e^{2\phi-\sigma}\left(dx^4 + B\right)^2 + e^{-2\phi-\sigma} ds^2_{(3)}. \tag{3}$$

Here, σ, ϕ are the Kaluza-Klein scalars; $\mathcal{A}^0 = \mathcal{A}^0_4 dx^4 + \mathcal{A}^0_m dx^m$ and $B = B_m dx^m$ are 1-forms corresponding to the Kaluza-Klein vectors; and $ds^2_{(3)}$ is the metric on the 3-dimensional 'transverse' space. The constants $\epsilon_{1,2}$ determine whether the reductions over $x^{0.4}$, respectively, are performed over a timelike or spacelike direction. They take values $+1$ for timelike and -1 for spacelike reductions.

In theory, then, we should be able to use this information to reduce our original action (1) over x^0 and x^4, to obtain a 3-dimensional action consisting of scalars, vectors, and a rank 2 symmetric tensor. Before we do this though, there is one more important point regarding antisymmetric representations of the Poincaré group. Namely, in D dimensions, a p-form is dual to a $(D-p-2)$-form. The details to show this are presented in Section 7.8 of [2]. The idea, though, is that a p-form A_p enters into the action through a $(p+1)$-form field strength F_{p+1}. The equations of motion read $d * F_{p+1} = 0$, where $*$ denotes the Hodge dual. This implies that, locally, we can write the $(D-p-1)$-form $*F_{p+1}$ as dA_{D-p-2}, for some $(D-p-2)$-form A_{D-p-2}, which we call the "dual" of A_p.

In particular, taking $D = 3, p = 1$, we see that 1-forms (i.e. vector fields) in 3 dimensions are dual to scalars. This means that we can replace all of our 3-dimensional vector fields with scalar fields in the final action.

3.3 The 3d action

The 3d action in terms of the real 5d variables.

We're now all set to follow through the steps in the calculation outlined above, and reduce our theory over the two isometric directions. We won't do anything explicitly here for sake of brevity (I have notes where it's done explicitly if you're interested), but let's just write down the 3-dimensional Lagrangian that we end up with, and then explain the notation.

$$
\begin{aligned}
\tilde{\mathcal{L}}_3 &= -\epsilon_1 \hat{g}_{ij}\partial x^i \partial x^j + \hat{g}_{ij}\partial y^i \partial y^j - (\partial\phi)^2 \\
&+\frac{1}{4}e^{-4\phi}\epsilon_1\left(\partial\tilde{\phi} + p^I \overleftrightarrow{\partial} s_I\right)^2 + \frac{1}{12}e^{-2\phi}\epsilon(cyyy)(\partial p^0)^2 \\
&-\frac{1}{3}e^{-2\phi}\epsilon_2(cyyy)\hat{g}_{ij}\left(\partial p^i - x^i\partial p^0\right)\left(\partial p^j - x^j \partial p^0\right) \\
&+3e^{-2\phi}\epsilon_2(cyyy)^{-1}\left(\partial s_0 + x^i\partial s_i - \frac{1}{6}(cxxx)\partial p^0 + \frac{1}{2}(cxx)_i\partial p^i\right)^2 \\
&-\frac{3}{4}e^{-2\phi}\epsilon(cyyy)^{-1}\hat{g}^{ij}\left(\partial s_i - \frac{1}{2}(cxx)_i\partial p^0 + (cx)_{ik}\partial p^k\right) \\
&\quad\times\left(\partial s_j - \frac{1}{2}(cxx)_j\partial p^0 + (cx)_{jl}\partial p^l\right). \qquad (4)
\end{aligned}
$$

Let's identify how each of the fields in (4) correspond to our original 5-dimensional fields. We have

$$
y^i \sim e^{\sigma}h^i, \quad x^i \sim \mathcal{A}^i_0, \quad p^i \sim \mathcal{A}^i_4 - \mathcal{A}^i_0\mathcal{A}^0_4, \quad p^0 \sim \mathcal{A}^0_4. \qquad (5)
$$

The dual scalars $\tilde{\phi}, s_0, s_i$ are related to the 3-dimensional vector fields B_m, $\mathcal{A}^0_m$, and $\mathcal{A}^i_m$ respectively, but the explicit expressions are somewhat involved. We'll write them down after we have truncated some of the fields, below.

The coupling $\hat{g}_{ij}(y)$ is given by

$$
\hat{g}_{ij}(y) = \frac{3}{2}\left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2}\frac{(cyy)_i(cyy)_j}{(cyyy)^2}\right), \qquad (6)
$$

and we note finally that the hypersurface constraint $\hat{\mathcal{V}}(h) = 1$ becomes $\hat{\mathcal{V}}(y) = 6e^{3\sigma}$.

3.4 Consistent truncations

Definition in general. Specific truncation for static, magnetic strings.

The action (4) looks a little unwieldy. We'd like to simplify our lives by only considering field configurations where some subset of the scalars are switched off. However, we can't just switch off scalars arbitrarily: we need to perform what is known as a "consistent truncation". The idea is that we can consistently truncate (i.e. set to zero) a field if, when we set it to zero on some initial time slice, it will remain zero on all future time slices. That is, we want the truncation to commute (in a suitable sense) with the equations of motion.

For the solutions we're interested in, we want to be able to describe static non-rotating magnetic black strings. This corresponds to a consistent truncation where we set to zero the Kaluza-Klein 1-forms $\mathcal{A}^0$ and B, and the electric charges $\mathcal{A}^i_0$ and $\mathcal{A}^i_4$. In terms of the Lagrangian (4) this corresponds to setting

$$
x^i = p^i = s_0 = p^0 = \tilde{\phi} = 0.
$$

Simultaneously redefining

$$w^i = e^{-\phi-\frac{3}{2}\sigma}y^i, \qquad \xi = \phi - \frac{3}{2}\sigma,$$

gives us the truncated 3-dimensional action

$$S_3 = \int d^3x\sqrt{g}\left[\frac{r}{2} + \hat{g}_{ij}(w)\partial w^i \partial w^j - \frac{1}{8}\hat{g}^{ij}(w)\partial s_i \partial s_j - \frac{1}{4}(\partial\xi)^2\right]. \qquad (7)$$

We note too that we have the relations

$$ds^2_{(5)} = e^{\xi+2\sigma}\left[-\epsilon_1 e^{-\xi}(dx^0)^2 - \epsilon_2 e^{\xi}(dx^4)^2\right] + e^{-2(\xi+2\sigma)}ds^2_{(3)}, \qquad (8)$$

for the metric, and

$$h^i = e^{\xi+2\sigma}w^i, \qquad F^i_{mn} = \frac{1}{4}\epsilon_{mnp}\hat{g}^{ij}(w)\partial^p s_j, \qquad (9)$$

for the remaining fields. We have at this point (implicitly) restricted to reductions which give rise to a Euclidean theory in 3 dimensions. Similar relations would hold if we reduced to a Lorentzian spacetime, but such cases are unimportant for our purposes.

4 Geometry of the target space

Where we give some detail as to how the moduli space determines the types of theories we can get.

Up until now, we have been interested solely in the geometry of our spacetime (be it 5- or 3-dimensional). We'll now introduce a second type of geometrical structure that is present in our theories: that of a scalar target space (or moduli space). Let's take as our first example the original 5d action (1) and isolate the part involving the scalar fields h^i

$$S_{\text{scalar}} = -\frac{3}{4}\int d^5x\sqrt{\hat{g}}\, a_{ij}(h)\partial_M h^i \partial^M h^j, \qquad (10)$$

where we recall that the indices i, j run over $1, \ldots, n_V + 1$. The form of the Lagrangian, where we have what looks like an ordinary scalar kinetic term, but with coupling a function of the scalars, is called a "non-linear sigma model" for historical pion-physics reasons.

If we work out the variation of (10) with respect to the fields h^i we find that the resulting equations of motion read

$$\Box h^i + \frac{1}{2}a^{ij}(h)\left(\partial_m a_{jn}(h) + \partial_n a_{mj}(h) - \partial_j a_{mn}(h)\right)\partial_M h^m \partial^M h^n = 0,$$

which we recognise as the equation for geodesic motion on a space with coordinates h^i and metric $a_{ij}(h)$! The space $(\mathcal{M}, a)$ is referred to by many names: the scalar manifold, the target space, the moduli space. We will probably slip in and out of using them all at some point.

4.1 r-, c-, and q-maps

VMs to HMs. Maps between manifolds.

For theories of $\mathcal{N} = 2$ supergravity coupled to vector or hypermultiplets, the scalar manifolds which appear take on a rather particular form: they define the so-called "special geometries" which OV talked about in his recent seminar. For a readable and fairly concise introduction, I'd recommend Section 20.3 of [2]. Here I'll just give a brief overview.

For the hypermultiplet sector of a theory (i.e. the manifold associated with the hypermultiplet scalars) we always get a quaternionic or para-quaternionic manifold. This decouples from the manifold of vector multiplets in the sense that the total scalar manifold is a direct product.

The manifold associated with 5-dimensional vector multiplets[5] is called "projective very special real"; that associated with 4-dimensional vector multiplets is called "projective special Kähler.

Dimensional reduction induces various maps between the scalar manifolds of these theories. Let's take a look at how this works.

Suppose we start with n_V 5-dimensional $\mathcal{N} = 2$ vector multiplets coupled to supergravity. So we have n_V real scalars which parametrise an n_V-dimensional projective special real manifold.

Under dimensional reduction from 5 to 4 dimensions, we get an extra $n_V + 1$ scalars from the 5-dimensional vectors (including the graviphoton) and one from the graviton. This gives us a total of $2(n_V + 1)$ scalars in the 4-dimensional theory, which we can package into $n_V + 1$ complex or para-complex[6] fields to complete our $n_V + 1$ 4-dimensional vector multiplets. The scalar manifold in this case is a very special (para-)Kähler manifold.

Hence, dimensional reduction from 5 to 4 dimensions induces a map from an n_V-dimensional very special real manifold to an $n_V + 1$-dimensional very special (para-)Kähler manifold. This is the r-map.

Similarly, dimensional reduction of our $n_V + 1$ 4-dimensional vector multiplets to 3 dimensions gives (after dualising the vectors) $4(n_V + 2)$ scalars which can be combined into $n_V + 2$ hypermultiplets. As we saw above, these parametrise a (para-)quaternionic manifold.

Dimensional reduction from 4 to 3 dimensions, in general, induces a map from special Kähler to special quaternionic manifolds, called the c-map. Reduction of those very special Kähler manifolds in the image of the r-map induces a further map (the composition of the r- and c-maps) between very special real and very special quaternionic manifolds, known as the q-map.

This subsection has, since it involves a slight digression from the main thread of our subject, been regrettably brief, especially regarding the notion of 'para'-complex geometry. For more information on the r-map, see [5]; for information on c-map, see [6]. The case of the supergravity q-map will be discussed at length

[5] When we restrict to global supersymmetry, replace 'projective' with 'affine'. See OV's talk for details on the difference.

[6] We will get complex fields if the reduction is performed over a spacelike direction, para-complex if it's performed over a timelike direction.

in a future publication.

5 Extremal black strings

Where we construct an extremal magnetic black string solution. Truncating a further scalar. Equations of motion.

Now that we've met all of the important players, we should let them get warmed up. To this end, we'll construct our first solution to the $\mathcal{N} = 2$, $d = (1, 4)$ supergravity theory described by the action (1): the extremal magnetic black string. For this, we truncate the scalar ξ from the action (7), metric ansatz (8), and fields (9), leaving us with an action

$$S_3 = \int d^3x\sqrt{g}\left[\frac{r}{2} + \hat{g}_{ij}(w)\partial w^i \partial w^j - \frac{1}{8}\hat{g}^{ij}(w)\partial s_i \partial s_j\right], \tag{11}$$

and a metric ansatz

$$ds^2_{(5)} = e^{2\sigma}\left(-\epsilon_1(dx^0)^2 - \epsilon_2(dx^4)^2\right) + e^{-4\sigma}ds^2_{(3)}, \tag{12}$$

where now the scalars w^i are related to the h^i as $w^i = e^{-2\sigma}h^i$. We will keep the form of the transverse metric $ds^2_{(3)}$ arbitrary for this example, which will allow us to construct multi-centred extremal black strings.

The Einstein equations, and the equations of motion for the scalars w^i and dual scalars s_i are, respectively,

$$\begin{aligned}
&\frac{1}{2}r_{mn} + \hat{g}_{ij}(w)\left[\partial_m w^i \partial_n w^j - \frac{1}{8}\hat{g}^{ik}(w)\partial_m s_k\, \hat{g}^{jl}(w)\partial_m s_l\right] = 0,\\
&\hat{g}_{ij}(w)\triangle w^j + \frac{1}{2}(\partial_i \hat{g}_{jk})\left[\partial w^j \partial w^k - \frac{1}{8}\hat{g}^{jl}(w)\partial s_l\, \hat{g}^{kr}(w)\partial s_r\right] = 0,\\
&\partial_m\left(\sqrt{g}\,\hat{g}^{ij}(w)\partial^m s_j\right) = 0,
\end{aligned} \tag{13}$$

where $\triangle$ denotes the Laplacian in the 3-dimensional transverse space. Our goal now is to find some configuration of the fields g_{mn}, w^i, s_i which satisfy (13). This will give us an instanton solution to our 3-dimensional theory (11). This can then be lifted to a solitonic solution of the original 5-dimensional theory (1).

5.1 Generalized extremal instanton ansatz

Generalized EIA. Mention multi-centred solutions. Explicit form for spherically symmetric.

One way of simplifying matters is to impose what is known as the generalized extremal instanton ansatz (GEIA) on (13). Suppose there exists a constant matrix $R^i{}_j$ satisfying $R^T\hat{g}R = \hat{g}$. We can certainly choose $R = \pm\mathbb{I}$, but in general there may be a number of other possible choices, depending on the model we're interested in. Then, the GEIA relates the scalars w^i and s_i through

$$\partial_m w^i = \frac{1}{\sqrt{8}} R^i{}_j \hat{g}^{jk}(w) \partial_m s_k. \tag{14}$$

We note now that, plugging this into (13), the equations of motion reduce to

$$r_{mn} = 0, \qquad \triangle w^i = 0. \tag{15}$$

In other words, the 3-dimensional transverse space is Ricci flat, and the w^i are harmonic functions H^i of the transverse coordinates.

Using (9) we can see that the field strength is given by

$$F^i_{mn} = \frac{1}{\sqrt{2}} \epsilon_{mnp} (R^{-1})^i{}_j \partial^p H^j. \tag{16}$$

whereas the scalars σ are given by $\hat{\mathcal{V}}(H) = e^{-6\sigma}$. This allows us to write the general form of an extremal magnetic black string solution to (1) as

$$ds^2_{(5)} = \hat{\mathcal{V}}(H)^{-\frac{1}{3}} (-dt^2 + dy^2) + \hat{\mathcal{V}}(H)^{\frac{2}{3}} ds^2_{(3)}, \tag{17}$$

where we have denoted by (t, y) the timelike and spacelike directions of the string's worldvolume.

Before moving on, let's restrict attention to the case where the transverse space is spherically symmetric,

$$ds^2_{(3)} = d\rho^2 + \rho^2 d\Omega^2_2.$$

In this case, taking into account (16), we see that we can write the harmonic functions as

$$H^i(\rho) = h^i_\infty + \frac{R^i{}_j \tilde{p}^j}{\rho}, \tag{18}$$

where we have introduced the physical parameters (moduli) h^i_∞, corresponding to the asymptotic value of the scalars $h^i(\rho)$, and $\tilde{p}^i$, corresponding to the physical magnetic charges.

We note that, in order that we really are describing a regular black string with horizon at $\rho = 0$, there should be no zeroes in $\hat{\mathcal{V}}(H)$ for $\rho > 0$. In terms of the moduli, this tells us that we must have

$$\mathrm{sign}(h^i_\infty) = \mathrm{sign}(R^i{}_j \tilde{p}^j). \tag{19}$$

We will use this condition in the following subsection, where we focus on a particular choice of the prepotential $\hat{\mathcal{V}}(h)$.

5.2 Example: ST^2 model

Choice of prepotential. 1-dimensional moduli space. Choices of extremal instanton ansatz.

Let us now turn our attention to a specific model, i.e. a specific choice of prepotential $\hat{\mathcal{V}}(h)$. We'll consider the so-called ST^2 model, which describes the 1-dimensional special real manifold $h^0(h^1)^2 = 1$. That is, we take as prepotential

$$\hat{\mathcal{V}}(h^0, h^1) = h^0(h^1)^2.$$

In order for the hypersurface to be well-defined, we see that we need $h^0 > 0$. There are then two disjoint patches in which h^1 can take values, namely $\{h^1 > 0\}$ and $\{h^1 < 0\}$. Working out the associated metric $\hat{g}_{ij}$, we find

$$\hat{g}_{ij} \sim \begin{pmatrix} (h^1)^4 & 0 \\ 0 & 2h^0 \end{pmatrix}.$$

It turns out that there are 4 possible "R-matrices" satisfying $R^T \hat{g} R = \hat{g}$, namely $R = \pm R_{(\sigma)}$, where

$$R_{(\sigma)} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix},$$

for $\sigma = \pm 1$.

For each choice of R-matrix we have a different field configuration. We will concentrate on those extremal solutions for which the 3-dimensional transverse space is spherically symmetric. In this case, the harmonic functions which characterise the solution are given by (18).

5.3 Aside: Tension and central charge

Calculating tension. Calculating central charge.

Let's take a brief hiatus and consider how we should go about calculating the tension (mass per unit length) and central charge of our solutions. We follow [7].

Let $x^{0,4}$ be the worldvolume coordinates of the black string, and consider the asymptotic metric in these directions, namely

$$g_{ab} = \eta_{ab} + \frac{c_{ab}}{\rho} + \mathcal{O}(\rho^{-2}),$$

for some constants c_{ab}. The tension is then given by

$$T := \frac{1}{4}\left(c_{00} - \frac{1}{4}\eta^{ab}c_{ab}\right) = \frac{1}{2}c_{00} - \frac{1}{4}c_{44}. \tag{20}$$

In the case of the metric (17) (where $x^0 = t$ and $x^4 = y$) and the ST^2 model we're considering, this reads

$$T = \frac{1}{4}\left[(h^1_\infty)^2 (R\tilde{p})^0 + 2h^0_\infty h^1_\infty (R\tilde{p})^1\right]. \tag{21}$$

For the central charge, we have the expression [7]

$$\mathcal{Z} = c_{ijk} h^i_\infty h^j_\infty \tilde{p}^k = \frac{1}{3}\left[(h^1_\infty)^2 \tilde{p}^0 + 2 h^0_\infty h^1_\infty \tilde{p}^1\right]. \tag{22}$$

5.4 BPS/non-BPS regions of moduli space

Identifying BPS/non-BPS regions of moduli space. Choices of R-matrix.

We are now in a position to understand the rôle of the R-matrix in our solutions. Let's go through each choice one-by-one and see what we find. We have to bear in mind two things: the hypersurface should be well-defined ($h^0_\infty > 0$) and the solution should be regular (19).

For the case $R = +R_1$, we have $\tilde{p}^0 > 0$ and $\mathrm{sign}(h^1_\infty) = \mathrm{sign}(\tilde{p}^1)$. The tension is given by

$$T = \frac{1}{4}\left[(h^1_\infty)^2 \tilde{p}^0 + 2 h^0_\infty h^1_\infty \tilde{p}^1\right].$$

For the case $R = -R_1$, we have $\tilde{p}^0 < 0$ and $\mathrm{sign}(h^1_\infty) = -\mathrm{sign}(\tilde{p}^1)$. The tension is given by

$$T = -\frac{1}{4}\left[(h^1_\infty)^2 \tilde{p}^0 + 2 h^0_\infty h^1_\infty \tilde{p}^1\right].$$

For the case $R = +R_{-1}$, we have $\tilde{p}^0 > 0$ and $\mathrm{sign}(h^1_\infty) = -\mathrm{sign}(\tilde{p}^1)$. The tension is given by

$$T = \frac{1}{4}\left[(h^1_\infty)^2 \tilde{p}^0 - 2 h^0_\infty h^1_\infty \tilde{p}^1\right].$$

For the case $R = -R_{-1}$, we have $\tilde{p}^0 < 0$ and $\mathrm{sign}(h^1_\infty) = \mathrm{sign}(\tilde{p}^1)$. The tension is given by

$$T = -\frac{1}{4}\left[(h^1_\infty)^2 \tilde{p}^0 - 2 h^0_\infty h^1_\infty \tilde{p}^1\right].$$

We notice that, for $R = \pm R_1$, the tension and central charge satisfy the BPS condition $T = \frac{3}{4}|\mathcal{Z}|$, whereas this does not hold for $R = \pm R_{-1}$, which correspond to non-BPS states. This is a generic feature: taking an R-matrix proportional to the identity produces extremal BPS states, whilst $R \neq \pm \mathbb{I}$ corresponds to extremal non-BPS states.

In terms of the moduli space, then, we have the following situation. If the magnetic charges $\tilde{p}^0$ and $\tilde{p}^1$ have the same sign, then the "BPS region" is $\{h^1_\infty > 0\}$, whilst the "non-BPS region" is $\{h^1_\infty < 0\}$. If $\tilde{p}^0$ and $\tilde{p}^1$ are of opposite sign, then the BPS and non-BPS regions are exchanged.

6 Non-extremal black strings

Where we construct a non-extremal magnetic black string solution.

Now that we're all warmed-up and have a better idea of the methodology for constructing solutions, let's continue to the main act: the construction of non-extremal magnetic black string solutions to $\mathcal{N}=2$, $d=(1,4)$ supergravity coupled to vector multiplets. Although the work here is new, it is a fairly simple extension of [8], where they consider the case of non-extremal black hole solutions.

6.1 Spherically-symmetric line element

Spherically symmetric line element in 3d. Coordinates.

For the non-extremal solutions, we're going to be concentrating on those for which the transverse space is spherically symmetric. Moreover, we want to impose that our scalars w^i, s_i, ξ depend only on a single radial coordinate. The general ansatz for a spherically symmetric line element in 3 dimensions is

$$ds^2_{(3)} = e^{4A(\tau)}d\tau^2 + e^{2A(\tau)}\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{23}$$

The coordinate τ is an affine coordinate on the transverse space. The asymptotic region is at $\tau \to 0$, whereas the outer horizon is situated at $\tau \to \infty$. We can solve the Einstein equations derived from the action (7) with the metric ansatz (23) to find

$$ds^2_{(3)} = \frac{c^4}{\sinh^4(c\tau)}d\tau^2 + \frac{c^2}{\sinh^2(c\tau)}d\Omega^2_2, \tag{24}$$

where c is a constant (which we will later identify as a non-extremality parameter) which enters into the "Hamiltonian constraint[7]"

$$c^2 + \hat{g}_{ij}(w)\dot{w}^i\dot{w}^j - \frac{1}{8}\hat{g}^{ij}(w)\dot{s}_i\dot{s}_j - \frac{1}{4}\dot{\xi}^2 = 0, \tag{25}$$

where $\dot{w}$, etc. denotes differentiation with respect to τ.

It is useful to introduce the so-called isotropic coordinate $\tau \to \rho$ with

$$\rho = \frac{ce^{c\tau}}{\sinh(c\tau)}, \tag{26}$$

in which the line element (24) takes the form

$$ds^2_{(3)} = d\rho^2 + W(\rho)\rho^2 d\Omega^2_2, \tag{27}$$

where

$$W(\rho) = 1 - \frac{2c}{\rho} = e^{-2c\tau}. \tag{28}$$

So far we have only dealt with the Einstein equations for our theory. This has determined the form of the 3-dimensional transverse metric up to some

[7]This is just the $\tau\tau$ component of the Einstein equations, which is considered a constraint in the so-called "Hamiltonian formalism".

constant c. We still need to solve the equations of motion for the remaining scalar fields w^i, s_i, ξ subject to the Hamiltonian constraint (25). This we will do next.

6.2 Constructing the instanton solutions

Solving equations of motion for the scalars. Relation of scalars to physical degrees of freedom, i.e. charges, moduli, non-extremality parameter. Algebraic constraint.

Let us now set about solving the equations of motion for the scalar fields. For $\xi(\tau)$, this reads $\ddot{\xi} = 0$, which is solved by $\xi(\tau) = a\tau + b$. Inserting this into (8), we see that asymptotic flatness (recall that this corresponds to the $\tau \to 0$ limit) forces us to take $b = 0$.

Let's now think about the near-horizon region ($\tau \to \infty$). We note that, for a black string, the horizon is of the form $\mathbb{R} \times S^2$, where the S^2 factor is described by the $d\Omega_2^2$ piece in (24) and the $\mathbb{R}$ piece is the spatial direction along the worldvolume of the string. This can correspond to either the dx^0 or dx^4 piece in (8) depending on whether we have performed first a reduction over space or time. In either case, the condition on $\xi(\tau)$ is that, when we integrate over the S^2 and some finite piece of $\mathbb{R}$ to find the size of (a section of) the horizon, we should get a finite answer. This then imposes that

$$\xi(\tau) = \epsilon_1 c\tau. \tag{29}$$

Let us now move on to the dual scalars $s_i(\tau)$. The equations of motion for these are solved by

$$\dot{s}_i = \sqrt{8}\, \hat{g}_{ij}(w)\tilde{p}^j, \tag{30}$$

where the factor of $\sqrt{8}$ is chosen for later convenience and $\tilde{p}^i$ are constant. Relating this back to the field strengths, we find the only non-zero component to be

$$F^i_{\theta\varphi} = \frac{1}{\sqrt{2}}\epsilon_1 \tilde{p}^i \sin\theta, \tag{31}$$

which identifies the $\tilde{p}^i$ as the physical magnetic charges of the string.

Before we move on to consider the equations of motion for w^i, let us update ourselves on the current form of the Hamiltonian constraint (25), which now reads

$$\frac{3}{4}c^2 + \hat{g}_{ij}(w)\left(\dot{w}^i\dot{w}^j - \tilde{p}^i\tilde{p}^j\right) = 0. \tag{32}$$

Let's now move onto the equations of motion for w^i, which read

$$\hat{g}_{ij}\ddot{w}^j + \frac{1}{2}\partial_i\hat{g}_{jk}\left(\dot{w}^j\dot{w}^k - \tilde{p}^j\tilde{p}^k\right) = 0. \tag{33}$$

Contracting this with w^i and using various identities of special geometry, we arrive at the condition

$$\hat{g}_{ij}(w)w^i\left(\ddot{w}^j - c^2 w^j\right) = 0, \tag{34}$$

which can be solved by

$$\ddot{w}^i - c^2 w^i = X^i, \tag{35}$$

for some X^i satisfying $\hat{g}_{ij}(w)w^iX^j = 0$. For simplicity, we will look for solutions with $X^i = 0$. This will enable us to find explicit solutions for $w^i(\tau)$ for the so-called "block diagonal" models. It is possible that, by taking $X^i \neq 0$ we can find solutions for a more general class of models, but this work is yet to be carried out.

For the class of solutions with $X^i = 0$, (35) is solved by

$$w^i(\tau) = A^i \cosh(c\tau) + \frac{B^i}{c}\sinh(c\tau), \tag{36}$$

where we have chosen the coefficients in such a way as to recover the solution for the case of extremal spherically symmetric black strings, $w^i(\tau) = A^i + B^i\tau$, in the extremal limit $c \to 0$.

We can write the solution (36) in terms of the isotropic coordinate ρ introduced in (26) to find

$$w^i(\rho) = \left(A^i + \frac{p^i}{\rho}\right)W^{-\frac{1}{2}} := H^i(\rho)W^{-\frac{1}{2}}, \tag{37}$$

where we have used the definition of the function $W(\rho)$ given in (28), and introduced $p^i := B^i - cA^i$. At this point it's convenient also to introduce $\bar{p}^i := p^i + 2cA^i$. We will see later on that p^i and $\bar{p}^i$ are related to the values of the scalar fields $h^i(\rho)$ at the inner and outer horizons respectively.

Using (9), we see that

$$e^{\xi+2\sigma} = \hat{\mathcal{V}}(w)^{-\frac{1}{3}} = \hat{\mathcal{V}}(H)^{-\frac{1}{3}}W^{\frac{1}{2}},$$

so

$$h^i(\rho) = \hat{\mathcal{V}}(H)^{-\frac{1}{3}}H^i(\rho). \tag{38}$$

This tells us that, taking the asymptotic limit, we should have $A^i = h^i_\infty$.

We should also consider the limits $\rho \to 2c$ and $\rho \to 0$, which correspond to the outer and inner horizons respectively. We see that, in these cases, the scalars $h^i(\rho)$ satisfy

$$h^i \xrightarrow[\rho\to 2c]{} \left(\hat{\mathcal{V}}(\bar{p})(2c)^{-3}\right)^{-\frac{1}{3}}\frac{\bar{p}^i}{2c} = \hat{\mathcal{V}}(\bar{p})^{-\frac{1}{3}}\bar{p}^i,$$

and

$$h^i \xrightarrow[\rho\to 0]{} \left(\hat{\mathcal{V}}(p)(\rho)^{-3}\right)^{-\frac{1}{3}} \frac{p^i}{\rho} = \hat{\mathcal{V}}(p)^{-\frac{1}{3}} p^i.$$

This is the same "dressed attractor behaviour" as [8], and motivates calling $\bar{p}^i$ and p^i the outer and inner "horizon charges" respectively.

We still need to make sure that this field configuration satisfies the Hamiltonian constraint (32), which now corresponds to choosing constants $p^i, \bar{p}^i = p^i + 2ch^i_\infty, c$ such that

$$\hat{g}_{ij}(w)\left(\tilde{p}^i\tilde{p}^j - p^i\bar{p}^j\right) = 0. \tag{39}$$

6.3 Solution for diagonal models

Explicit relation to physical charges. Instanton solution. Lifting to 5d black string.

Although we're more interested in the general structure of non-extremal black strings, it wouldn't hurt to do things explicitly for a particularly simple class of models so that we can see how everything comes together.

Let's consider the so-called "diagonal models", where $\hat{g}_{ij}$ is diagonal. Then (39) is solved by taking

$$p^i = -ch^i_\infty \pm \sqrt{(\tilde{p}^i)^2 + c^2(h^i_\infty)^2},$$

for each $i = 1, \ldots, n_V + 1$. The sign must be chosen such that the solution is regular for $\rho > 0$, which corresponds to choosing the positive sign for $h^i_\infty > 0$ and the negative sign for $h^i_\infty < 0$.

Bringing everything together, we find

$$ds^2_{(5)} = \hat{\mathcal{V}}(H)^{-\frac{1}{3}}\left(-W\,dt^2 + dy^2\right) + \hat{\mathcal{V}}(H)^{\frac{2}{3}}\left(\frac{d\rho^2}{W} + \rho^2 d\Omega_2^2\right), \tag{40}$$

with

$$H^i(\rho) = h^i_\infty + \frac{p^i}{\rho}, \qquad p^i = -ch^i_\infty \pm \sqrt{(\tilde{p}^i)^2 + c^2(h^i_\infty)^2}, \tag{41}$$

and

$$F^i = \frac{1}{\sqrt{2}}\tilde{p}^i \sin\theta\, d\theta \wedge d\varphi. \tag{42}$$

A similar analysis can be performed for the block diagonal models considered in [8].

7 Relation between extremal and non-extremal solutions

Where we elucidate certain relationships between the non-extremal and extremal BPS/non-BPS black strings.

So we've now managed to describe both extremal and non-extremal magnetic black string solutions to our $\mathcal{N} = 2$, $d = (1,4)$ supergravity theory coupled to n_V vector multiplets. On the extremal side, we've seen that, depending on the model (i.e. choice of prepotential) we can have both BPS and non-BPS solutions.

On the non-extremal side, everything is fixed except for the relation between the horizon charges p^i and the physical magnetic charges $\tilde{p}^i$, which is encoded in the Hamiltonian constraint (39).

Let's consider the extremal $c \to 0$ limit of (39). In this limit, the inner and outer horizons coalesce, and we have $\bar{p}^i = p^i$, so that (39) becomes

$$\hat{g}_{ij}(w)\tilde{p}^i\tilde{p}^j = \hat{g}_{ij}(w)p^ip^j.$$

This can be solved by taking $p^i = R^i{}_j\tilde{p}^j$ for any matrix R satisfying $R^T\hat{g}R = \hat{g}$, which is precisely the relation between the horizon charges p^i and magnetic charges $\tilde{p}^i$ for the extremal case. Thus, in a sense, we can say that the extremal BPS and extremal non-BPS regions of moduli space can be reached by taking a suitable extremal limit of the non-extremal black string solution.

8 Small black holes in 4 dimensions

Small black holes as time-lifts of extremal instantons.

Before we finish, there's just enough room to look at a further class of solution which we get for free in this picture: a 4-dimensional small black hole.

Suppose we start off by reducing our original theory first over space and then over time. We have been able to find both extremal and non-extremal instanton solutions to the corresponding 3-dimensional Euclidean theory.

Take one of these extremal solutions and lift it over the timelike direction. This gives us a solitonic solution to the 4-dimensional Lorentzian theory. In particular, we find the 4-dimensional line element

$$ds^2_{\mathrm{SBH}} = -\hat{\mathcal{V}}(H)^{-\frac{1}{2}}dt^2 + \hat{\mathcal{V}}(H)^{\frac{1}{2}}ds^2_{(3)}. \tag{43}$$

In the case where the 3-dimensional transverse space is spherically symmetric, have a black hole with horizon at $\rho \to 0$. However, calculating the area of this horizon, we see that it vanishes. This identifies (43) as a so-called "small black hole".

9 Generalisations and future work

Generalized prepotentials.

Throughout the work so far we have restricted ourselves to prepotentials $\hat{\mathcal{V}}(h)$ which are polynomials of degree 3, described by coefficients c_{ijk}. This fact ensures that our action (1) is gauge invariant and supersymmetric [9]. However, we can also consider [10] a non-supersymmetric action described by a generalized prepotential which we take to be a polynomial of degree $p > 3$. The subtlety here is that we then lose gauge invariance of the Chern-Simons part of the action (1), since the coefficient multiplying it will be the third derivative of the prepotential and therefore non-constant. This can be avoided if we consider purely-electric or purely-magnetic theories, for which the Chern-Simons term doesn't contribute.

We can then follow the same procedure as above and construct non-SUSY magnetic black string solutions, which have a similar form to those found above.

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