Coulomb Branch and the Moduli Space of Instantons

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Based on work done in collaboration with
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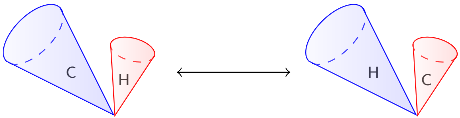
Introduction and Motivation

- Supersymmetric Gauge Theories in 3d with $\mathcal{N} = 4$ are subject to a peculiar duality: **Mirror Symmetry**
- 3d mirror symmetry exchanges **Coulomb branch** and **Higgs branch** of two dual theories.



- 3 pieces of jargon that become confusing under mirror symmetry:
  - Coulomb branch: moduli space parametrised by scalars in the V-plet
  - Higgs branch: moduli space parametrised by scalars in the H-plet
  - Moduli Space of Instantons

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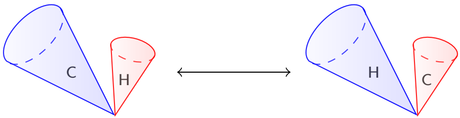

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- Goal:

Moduli Space
of G-instanton

$\cong$

Higgs branch
of specific theory

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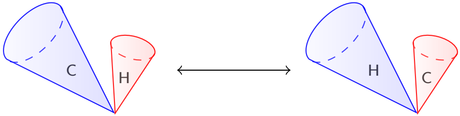

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$$\begin{array}{ccc} & \text{Moduli Space of G-instanton} & \\ \cong & & \\ \text{Higgs branch of specific theory} & \xleftrightarrow[\text{dual}]{} & \text{Coulomb branch of dual theory} \end{array}$$

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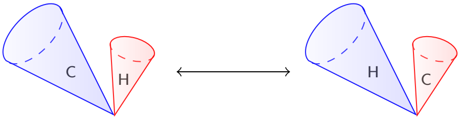

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$\cong$ $\cong$

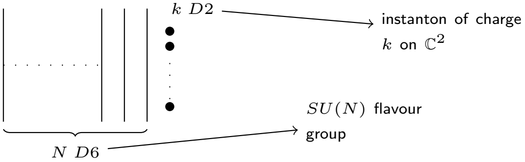
Higgs branch of specific theory $\longleftrightarrow$ Coulomb branch of dual theory

dual

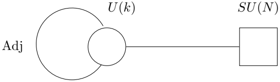
Outline

ADHM quivers

- Instantons have a well-known realization in terms of branes:
  Dp-branes inside D(p+4)-branes with or without O(p+4)-planes (orientifolds)
- To realize a 3d theory choose $p = 2$
  $\Rightarrow$ D2 branes in the background of D6-branes

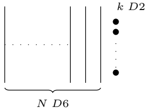
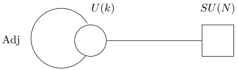


- This brane construction can be associated to a quiver gauge theory



- The Higgs branch of this quiver gauge theory is isomorphic to the moduli space of $k\ SU(N)$ instantons on $\mathbb{C}^2$
- To engineer other groups need a background with orientifolds

ADHM quivers

<table>
<tr><th>G</th><th>Brane configurations</th><th>ADHM quiver</th></tr>
<tr><td>A_{N-1}</td><td>
</td><td>
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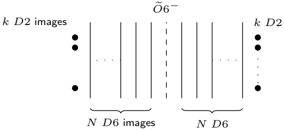
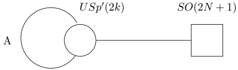
<tr><td>B_N</td><td>
</td><td>
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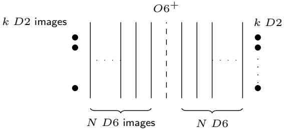
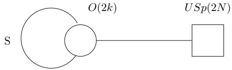
<tr><td>C_N</td><td>
</td><td>
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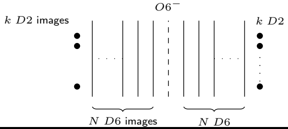
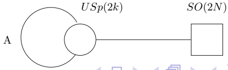
<tr><td>D_N</td><td>
</td><td>
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Hilbert series for Higgs branch

Higgs branch of ADHM quiver theories $\cong$ the moduli space of k G-instantons on $\mathbb{C}^2$

$$\mathcal{M}_{\mathsf{H}}^{\mathsf{ADHM}} \cong \widetilde{\mathcal{M}}_{\mathsf{k,G}} \qquad \text{on } \mathbb{C}^2$$

To study moduli space of G-instantons $\Rightarrow$ calculate Hilbert Series for the Higgs branch.

- What is the Hilbert Series (HS)?
  - It is a partition function that counts chiral gauge invariant operators
- Why do we care?
  - The chiral gauge invariant operators parametrise the moduli space
  - Hilbert Series encodes: dimension of moduli space, generators, relations

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- How do we calculate it?

HS for the space $\mathbb{C}^2/\mathbb{Z}_2$

- $\mathbb{C}^2$ with action of $\mathbb{Z}_2$: $(z_1, z_2) \longleftrightarrow (-z_1, -z_2)$
- Holomorphic functions invariant under this action: z_1^2, z_2^2, $z_1 z_2$, z_1^4, ...
- All monomials constructed from 3 generators subject to 1 relation
  - $X = z_1^2$, $Y = z_2^2$, $Z = z_1 z_2$
  - $XY = Z^2$

Hilbert series for Higgs branch

HS for the space $\mathbb{C}^2/\mathbb{Z}_2$

Collect the infinitely many invariants in 1 function

- Isometry group of $\mathbb{C}^2$: $U(2)$
- Cartan subalgebra: $U(1)^2$
- Choose counters or **fugacities** t_1, t_2
  - t_1 is the $U(1)$ charge of z_1
  - t_2 " " z_2

$$\mathrm{HS}(t_1,t_2) = 1 + t_1^2 + t_2^2 + t_1 t_2 + \ldots = \sum_{i,j=0}^{\infty} t_1^i t_2^j \quad \text{with } j = i \text{mod} 2$$

Hilbert series for Higgs branch

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- Can unrefine: $t_1 = t_2 = t \longrightarrow$ count all monomials at given degree

$$\mathrm{HS}(t) = \sum_{i,j\ldots} t^{i+j} = 1 + 3t + 5t^2 + \ldots = \sum_{k=0} (2k+1) t^{2k}$$

$$= \frac{1 - t^4}{(1 - t^2)^3}$$

- Dimension of moduli space = pole of unrefined HS

Hilbert series for Higgs branch

HS for the space $\mathbb{C}^2/\mathbb{Z}_2$

- Can refine: $t_1 = yt$ and $t_2 = t/y$

$$\mathrm{HS}(t;y) = 1 + (y^2+1+y^{-2})t^2 + (y^4+y^2+1+y^{-2}+y^{-4})t^4 + \ldots$$

$$= \sum_{k=0} \chi[2k]^{\mathsf{SU(2)}}_{\vec{y}} t^{2k}$$

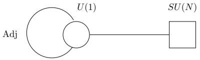
$$= \frac{1-t^4}{(1-t^2y^2)(1-t^2)(1-t^2y^{-2})}$$

  - where $y^2+1+y^{-2} = \chi[2]^{\mathsf{SU(2)}}_{\vec{y}}$
    $y^4+y^2+1+y^{-2}+y^{-4} = \chi[4]^{\mathsf{SU(2)}}_{\vec{y}}$

- generators: triplet of $SU(2) \longrightarrow\ X,\ Y,\ Z$ at degree 2
- relation: numerator $\to$ quadratic in generators

Hilbert series for Higgs branch

- $\mathbb{C}^2/\mathbb{Z}_2$ turns out to be $\widetilde{\mathcal{M}}_{1,\mathrm{SU(2)}}$
- From ADHM quiver



$$\mathrm{HS}_{1,\mathrm{SU(N)}}(t;\ x,\vec{y}) = \frac{1}{(1-tx)(1-tx^{-1})}\sum_{k=0}\chi[k,0,\dots,0,k]^{\mathrm{SU(N)}}_{\vec{y}}\ t^{2k}$$

  - factor outside sum $\rightarrow$ centre of the instanton on $\mathbb{C}^2$
  - sum $\rightarrow$ reduced moduli space $\widetilde{\mathcal{M}}_{1,\mathrm{SU(N)}}$
  - $\chi[1,0,\dots,0,1]^{\mathrm{SU(N)}}_{\vec{y}}$ = character of adjoint rep of $SU(N)$
  - Global symmetry $SU(2)_g \times SU(N)$ explicit

Hilbert series for Higgs branch

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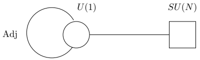

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  - Global symmetry $SU(2)_g \times SU(N)$ explicit
- Minimal nilpotent orbit of $SU(N)$

Generalise for any group G [Kronheimer]

$$\mathrm{HS}(\widetilde{\mathcal{M}}_{1,\mathrm{G}};t,\vec{y}) = \sum_{k=0}^{\infty}\chi[k\theta]^{\mathrm{G}}_{\vec{y}}\ t^{2k}$$

where $[\theta]^{\mathrm{G}}_{\vec{y}}$ is the adjoint rep of G

Hilbert series for Higgs branch

Difficulties

- For $k > 1$, no simple description of $\widetilde{\mathcal{M}}_{\mathsf{k,G}}$
- ADHM construction not known for exceptional groups
- HS for $k > 1$ from Higgs branch $\rightarrow$ HARD

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Solution: Mirror Symmetry

- Exchanges Coulomb branch with Higgs branch
- Find dual theory (use brane and dualities)
- Compute Coulomb branch HS
  - Coulomb branch HS receives quantum corrections
  - Unknow how to compute calculate HS until 2013 [Cremonesi, Hanany, Zaffaroni]

Dualities on brane construction

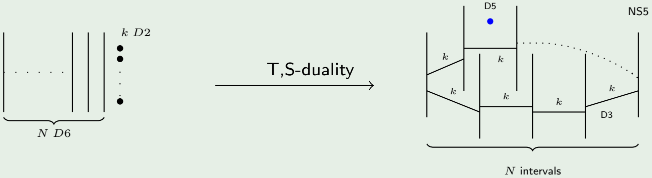
T-duality: D6-branes $\rightarrow$ D5-branes

D2-branes $\rightarrow$ D3-branes on a circle

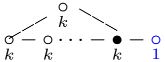
S-duality: D5-branes $\rightarrow$ NS5-branes

D3-branes $\rightarrow$ unchanged

No Orientifold



Necklace quiver:



Why do we do this?

- S-duality implements mirror symmetry
- S-duality on brane configurations $\rightarrow$ dual quiver gauge theory

Overextended Quivers

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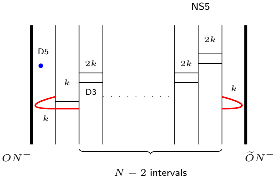
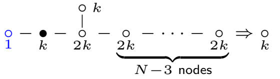
<tr><td>B_N</td><td>
</td><td>
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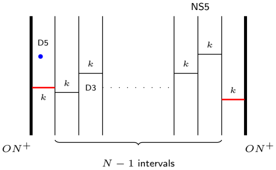
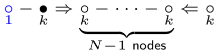
<tr><td>C_N</td><td>
</td><td>
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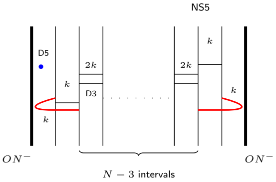
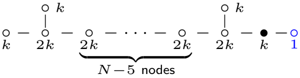
<tr><td>D_N</td><td>
</td><td>
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Overextended Quivers

Recap

- Start with ADHM quivers
- Do mirror symmetry to get dual theory
- Get nice fancy quivers $\boxed{\text{OVEREXTENDED DYNKIN DIAGRAMS}}$
  - Start with Dynkin diagram
  - Attach a node to the "adjoint node"
  - Attach another node $\Rightarrow$ overextended

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Extrapolate

G	Coulomb branch quivers
E_6	$\begin{array}{ccccccccccc} &&&&&& \circ\, k &&&& \\ &&&&&& \vert &&&& \\ &&&&&& \circ\, 2k &&&& \\ &&&&&& \vert &&&& \\ \color{blue}{\circ} & - & \bullet & - & \circ & - & \circ & - & \circ & - & \circ \\ \color{blue}{1} && k && 2k && 3k && 2k && k \end{array}$
E_7	$\begin{array}{ccccccccccccccc} &&&&&&&& \circ\, 2k &&&&&& \\ &&&&&&&& \vert &&&&&& \\ \color{blue}{\circ} & - & \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ \color{blue}{1} && k && 2k && 3k && 4k && 3k && 2k && k \end{array}$
E_8	$\begin{array}{ccccccccccccccccc} &&&&&&&&&&&& \circ\, 3k &&&& \\ &&&&&&&&&&&& \vert &&&& \\ \color{blue}{\circ} & - & \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ \color{blue}{1} && k && 2k && 3k && 4k && 5k && 6k && 4k && 2k \end{array}$
F_4	$\begin{array}{ccccccccccc} \color{blue}{\circ} & - & \bullet & - & \circ & - & \circ & \Rightarrow & \circ & - & \circ \\ \color{blue}{1} && k && 2k && 3k && 2k && k \end{array}$
G_2	$\begin{array}{ccccccc} \color{blue}{\circ} & - & \bullet & - & \circ & \Rrightarrow & \circ \\ \color{blue}{1} && k && 2k && k \end{array}$

Fields

<table>
<tr><th>$\mathcal{N} = 4$</th><th>$\mathcal{N} = 2$</th><th>Field(bosonic)</th><th>Label</th><th>$SU(2)_C$</th></tr>
<tr><td rowspan="3">V-plet
in adj of $\mathcal{G}$</td><td rowspan="2">V-plet: V</td><td>gauge</td><td>A_μ</td><td rowspan="3">$\vec{\phi} \equiv (\eta, \mathrm{Re}\Phi, \mathrm{Im}\Phi)$</td></tr>
<tr><td>real scalar</td><td>η</td></tr>
<tr><td>Chiral Φ</td><td>complex scalar</td><td>Φ</td></tr>
</table>

- $\langle \vec{\phi} \rangle \neq 0 \Rightarrow \mathcal{G} \to U(1)^r$, i.e left with photons
- photon in 3d dual to a scalar

$$F_{\mu\nu} = \epsilon_{\mu\nu\rho} \partial^\rho \gamma$$

- If some of the scalar VEV=0 $\Rightarrow A_\mu$ remains nonabelian
  - dualization not clear
- $\boxed{\text{Replace } (A_\mu, \eta) \to \text{monopole operators } V_m}$
  - Keep Φ
- $\Longrightarrow V_m$ and Φ parametrise $\mathcal{M}_C$

Monopole operators

What are monopole operators?

- Disorder operators inserted at x s.t A_μ has Dirac singularity at x

$$A^{N,S}(\vec{r}) = \frac{\mathbf{M}}{2}(\pm 1 - \cos\theta)\mathrm{d}\phi \qquad \mathbf{M} = \mathrm{diag}(m_1, ..., m_r)$$

- Dirac quantisation

$$\frac{e\mathbf{M}}{2\pi} \in \frac{\Lambda_w(\mathcal{G}^V)}{W}$$

  - where $\Lambda_w(\mathcal{G}^V)$ weight lattice of dual gauge group
  - W is the Weyl group

Monopole operators

Example: $U(N)$ monopole operators

- $\mathcal{G}^V = U(N)$

$$\Lambda_w(U(N)) = \mathbb{Z}^N$$

$$\Rightarrow m_i \in \mathbb{Z} \qquad i = 1, ..., N$$

- $W_{U(N)} = S_N$
  - lattice restricted to Weyl chamber

$$m_1 \geq m_2 \geq ... \geq m_N$$

Monopole operators

A crucial quantum number of monopole operators:

- The charge under $U(1)_C \subset SU(2)_C$ R-symmetry: $\Delta(\vec{m})$

$$\Delta(\vec{m}) = -\sum_{\vec{\alpha}\in\Delta_+} |\vec{\alpha}(\vec{m})| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\vec{\rho}_i\in\mathcal{R}_i} |\vec{\rho}_i(\vec{m})| \ ,$$

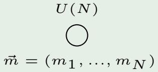
- first term: sum over the positive roots of $\mathcal{G}$
  - contribution from the gauge sector
- second term: sum over the weights of the reps of the hypers
  - contribution from the matter sector
- In IR $\boxed{\Delta(\vec{m}) = \text{scaling dimension of operators}}$

Monopole operators

Example: quivers with $U(N)$ nodes

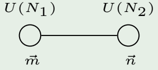
Two contributions:

- node



$$\Delta_{\text{vec}}(\vec{m}) = -\sum_{1\le i<j\le N} |m_i - m_j| \ .$$

- edge



$$\Delta_{\text{hyp}}(\vec{m},\vec{n}) = \frac{1}{2}\sum_{j=1}^{N_1}\sum_{k=1}^{N_2} |m_j - n_k|$$

- $\Delta_{\text{tot}}(\vec{m},\vec{n}) = -\sum_{i<j}^{N_1}|m_i-m_j| - \sum_{i<j}^{N_2}|n_i-n_j| + \frac{1}{2}\sum_{j=1}^{N_1}\sum_{k=1}^{N_2}|m_j-n_k|$

The Coulomb branch formula for HS

- Count monopole operators, grading them by their scaling dimension $\Delta(\mathbf{M})$

$$\mathrm{HS}(t) \sim \sum_{\Lambda_w(\mathcal{G}^V)/W} t^{2\Delta(\mathbf{M})}$$

- One important subtlety: need an extra factor in the sum!
  - The complex scalar Φ can still take VEV
  - It must be restricted:

$$\Phi_m \in \mathfrak{h}_m$$

  - H_m is the residual gauge group left unbroken by the monopole flux
- Really

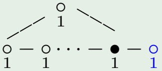
$$\mathrm{HS}(t) = \sum t^{2\Delta(\mathbf{M})} P_{\mathcal{G}}(t, \mathbf{M})$$

where $P_{\mathcal{G}}(t, \mathbf{M}) = \prod_{i=1}^{r} \frac{1}{1 - t^{2d_i(\mathbf{M})}}$

The Coulomb branch formula for HS

Example: 1-$SU(N)$ instanton using Coulomb branch

- Take the necklace quiver



<u>N nodes + overextended node</u>

overextended node (blue): m_1

affine node (black): m_0

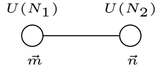
other nodes: m_i $i = 1, ...N-1$

- $\Delta = \frac{1}{2} \sum_{i=0}^{N-1} |m_i - m_{i+1}| + \frac{1}{2} |m_0 - m_{-1}|$
- Need to set $m_{-1} = 0 \rightarrow$ decouple overall $U(1)$

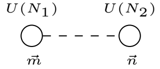
$$\boxed{\begin{aligned} \mathrm{HS}_{1,N}(t) &= \sum_{m_0=0}^{\infty} \cdots \sum_{m_{N-1}=0}^{\infty} t^{2\Delta(\mathbf{M})} \frac{1}{(1-t^2)^{N-1}} \\ &= \frac{1}{(1-t)^2} \sum_{k_0=0}^{\infty} dim\left([k,0,...,0,k]_{SU(N)}\right) t^{2k} \end{aligned}}$$

Non simply laced quivers

- How do we deal with the quivers which are not simply laced (B_N, C_N, F_4, G_2)?
- Previously

	$\Delta_{\text{hyp}}(\vec{m}, \vec{n}) = \frac{1}{2} \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} \lvert m_j - n_k \rvert$

- How about the triple lace in $\underset{1}{\circ} - \underset{k}{\bullet} - \underset{2k}{\circ} \Rrightarrow \underset{k}{\circ}$?

	$\Delta_{\text{hyp}}(\vec{m}, \vec{n}) = \frac{1}{2} \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} \lvert \lambda m_j - n_k \rvert$

where $\lambda = 1, 2,$ or 3 for a single, double or triple bond respectively

- We can now explicitly compute the HS of exotic things like:

$\boxed{\text{Moduli space of 3 } G_2 \text{ instantons}}$

Summary and Conclusions

- $3d$ $\mathcal{N} = 4$ gauge theories enjoy a powerful symmetry that exchanges Higgs branch with Coulomb branch
- Identified the moduli space of instantons with Higgs branch of ADHM quivers
- Found dual theories using branes
- Studied the chiral ring on the Coulomb branch of $3d\mathcal{N} = 4$ using monopole operators
- We have a new prescription to calculate HS associated to the chiral *a*ny group G and charge k
- By refining the Hilbert series, can extract the generators of the moduli space (combination of bare and dressed monopole operators)
- Extracting all the relations is not easy $\rightarrow$ future work
- Generalise to instantons on other spaces, like ALE spaces

Thank you!