The a-theorem: Constructing the A-function for a general theory

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Outline

What is the a-theorem?

Consider a QFT with couplings g_I at energy scale μ.

- (Zam., 1986) For 2D QFT, $\exists\, c(\mu, g_I)$ decreasing monotonically under RG flow. At a fixed point g_I^*, $c(\mu, g_I^*) = c$, the central charge.
- (Cardy, 1988) What about 4D? Possible that $\exists\, a(\mu, g_I)$ with monotonic behaviour under RG flow (*Strong*), or $\exists\, a(\mu, g_I)$ satisfying $(a_{UV} - a_{IR}) > 0$ (*Weak*).
- Name a-theorem comes from only possible candidate: $\langle T^{\mu}_{\mu} \rangle = -\mathbf{c}R$ in 2D, $\langle T^{\mu}_{\mu} \rangle = cF - \frac{1}{4}\mathbf{a}G - \frac{1}{72}gR^2 + \ldots$ in 4D.

Why bother?

- Flowing towards IR, $a(\mu, g_I)$ should decrease and approach a new RG fixed point, defining a low-energy effective theory with less "degrees of freedom".
- Monotonic flow will help address the possibiliy of limit cycles or chaotic behaviour in RG flows, showing they cannot occur for a renormalizable 4D QFT.
- In 2D scale-invariance implies conformal invariance (Pol., 1988); a-theorem may give insight as to whether something similar holds in 4D.

What's been done?

- 1990: Jack and Osborn - criteria for the a-theorem; perturbative a-theorem for sufficiently weak coupling.
- 2004: Intriligator *et al* - explicit a for SUSY theories.
- 2011: Komargodski and Schwimmer - weak a-theorem using 4-dilaton amplitude.
- 2014: Jack and Osborn - how are perturbative quantities constrained?

The theoretical preamble

Take a 4D QFT with couplings $\{g^I\}$.

- $\exists A$ such that $\partial_I A = T_{IJ}\beta^J$.
- For RG fixed point g^*, $\beta^I(g^*) = 0$ and $\frac{1}{4}A = a$. For non-RG fixed points, A arbitrary up to $\beta^I H_{IJ}\beta^J$.
- $G_{IJ} = T_{(IJ)}$ satisfies $\mu\frac{d}{d\mu}A = \beta^I\partial_I A = \beta^I G_{IJ}\beta^J$; shows flow is monotonic.
- If $\exists$ positive-definite G_{IJ}, the strong a-theorem holds.
- Multiplying by dg^I gives $dA = dg^I T_{IJ}\beta^J$, i.e. first order differential equation: solve for A.

Non-supersymmetric theories

Consider a general renormalizable gauge theory with simple gauge group G, n_ϕ real scalars, n_ψ Weyl fermions. Couplings are $\{g^I\} = \{Y, \bar{Y}, \lambda, g\}$, scalar/fermion gauge generators are $t_A^\phi = -t_A^{\phi T}$, t_A^ψ respectively. Assemble matrices

$$y_a = \begin{pmatrix} Y_a & 0 \\ 0 & \bar{Y}_a \end{pmatrix}, \hat{y}_a = \begin{pmatrix} \bar{Y}_a & 0 \\ 0 & Y_a \end{pmatrix}, T_A = \begin{pmatrix} t_A^\psi & 0 \\ 0 & -t_A^{\psi *} \end{pmatrix},$$

$$\hat{T}_A = -T_A^T$$

- Assume $T_{IJ} = G_{IJ}$ is symmetric for simplicity (works for first three orders).
- Lowest order A is $A^{(2)}$: solve $d_g A^{(2)} = dg\ G_{gg}^{(1)}\ \beta_g^{(1)}$
- Next lowest is $A^{(3)}$: solve $d_g A^{(3)} = dg\ G_{gg}^{(1)} \beta_g^{(2)} + dg\ G_{gg}^{(2)}\ \beta_g^{(1)}$, $d_y A^{(3)} = dy\ G_{y\bar{y}}^{(2)} \beta_{\bar{y}}^{(1)}$

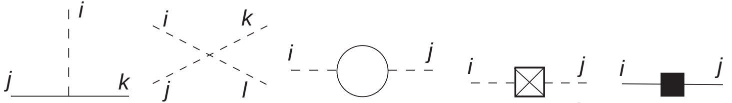
Non-supersymmetric theories

Diagrammatical notation

General expressions for β functions become long and complex:

$$\beta^{(1)}_{y\,a} = \frac{1}{2}(y_b\hat{y}_b - 6g^2\hat{C}^{\psi})y_a + \frac{1}{2}y_a(\hat{y}_b y_b - 6g^2 C^{\psi}) + \frac{1}{2}tr(y_a\hat{y}_b)y_b + 2\hat{y}_b y_a \hat{y}_b$$

One-loop Yukawa isn't too bad, but Two-loop has 29 terms...
Introduce Diagrammatical notation:



$$y_{ijk} \equiv y_i \qquad \lambda_{ij}{}^{kl} \qquad tr(y_i\bar{y}_j) \qquad g^2\, C^{\phi}{}_i{}^j \qquad g^2\, C^{\psi}{}_i{}^j$$

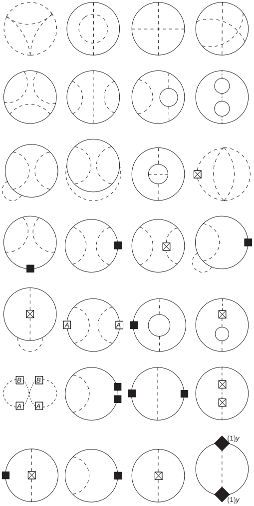
Non-supersymmetric theories
Solutions and constraints

For complete diagrammatic details, including definitions of β function coefficients, see *arXiv:1411.1301v1*.

- First non-trivial results are obtained from $A^{(4)}$, determined by $d_y A^{(4)} = dy\, G^{(2)}_{y\bar{y}} \beta^{(2)}_{\bar{y}} + dy\, G^{(3)}_{y\bar{y}}\ \beta^{(1)}_{\bar{y}}$ and $d_\lambda A^{(4)} = d\lambda\, G^{(3)}_{\lambda\lambda^*} \beta^{(1)}_{\lambda^*}$ (modulo purely gauge terms).
- $G^{(2)}_{y\bar{y}}$ and $G^{(3)}_{\lambda\lambda^*}$ consist of only one possible tensor structure each, but $G^{(3)}_{y\bar{y}}$ has 10 possible structures.
- Allowing β function and tensor structure coefficients to be arbitrary, $A^{(4)}$ can be solved for any renormalization scheme, up to $\beta^{(1)}_y \circ \beta^{(1)}_{\bar{y}}$, where $x \circ x = x^{ijk} x_{ijk}$.

Non-supersymmetric theories

Solutions and constraints



A solution of the ODEs produces linear equations relating the coefficients of $A^{(4)}$, $G^{(2)}$, $G^{(3)}$, β_y and β_λ. Eliminating all A and metric coefficients leaves 6 consistency conditions between coefficients in $\beta_y^{(2)}$.

Non-supersymmetric theories

Solutions and constraints

- Consistency conditions are independent of renormalization scheme.
- Change in scheme corresponds to coupling constant redefinition: change in $\beta^{(1)}$ induces change in $\beta^{(2)}$ via $\delta\beta_I^{(2)} = \left(\beta_J^{(1)} \frac{\delta}{\delta g^J}\right) \delta g_I - \left(\delta g_J \frac{\delta}{\delta g^J}\right) \beta_I^{(1)}$
- Consistency conditions invariant under arbitrary redefinition - must hold in all schemes.
- $d_g A^{(4)} = dg\, G_{gg}^{(3)} \beta_g^{(1)} + dg\, G_{gg}^{(2)} \beta_g^{(2)} + dg\, G_{gg}^{(1)} \beta_g^{(3)}$, hence can determine Yukawa/scalar terms in $\beta_g^{(3)}$ using only $\beta_g^{(1)}$ and $\beta_g^{(2)}$.

Supersymmetric theories

Supersymmetric theories work pretty much the same way, using the same structures G_{IJ} adapted to the supersymmetric case. For a general $\mathcal{N} = 1$ theory with a chiral (and anti-chiral) superfield,

- $\lambda \equiv \lambda(y, \bar{y}, g)$, so $\{g^I\} = \{y, \bar{y}, g\}$
- Anomalous dimension $\gamma^{(1)} = \frac{1}{2} y^{imn} \bar{y}_{mnj} - 2g^2 (T^2)^i{}_j$
- $\beta_y = (\gamma * y)^{ijk} \equiv (\gamma)^i{}_m y^{mjk} + (\gamma)^j{}_m y^{imk} + (\gamma)^k{}_m y^{ijm}$

$A^{(4)}$ has only 9 terms in the supersymmetric case, compared to the 28 terms in the non-supersymmetric case.

Supersymmetric theories
Does SUSY match Non-SUSY?

Although $A^{(4)}$ holds in the non-SUSY case regardless of scheme, one must use a SUSY-preserving scheme to compare results.

- Most convenient SUSY scheme is NSVZ since $\exists$ an exact β_g; NSVZ coincides with dimensional reduction (DRED) to two loops.
- DRED can be used for non-SUSY theories if treated properly, so the non-SUSY DRED $A^{(4)]}$ should reduce to the SUSY DRED $A^{(4)}$.
- Correctly defining SUSY y matrices to include gaugino fields, the results do indeed match.

Supersymmetric theories
The Λ-equation

There is a potential, nonperturbative A candidate for SUSY gauge theories with n_c chiral scalar multiplets:

$$A = \frac{1}{12}(n_c + 9n_v) - \frac{1}{2}tr(\gamma)^2 + \frac{1}{3}tr(\gamma)^3 + \Lambda \circ \beta_{\bar{y}} + \beta_y \circ H \circ \beta_{\bar{y}} + n_v \lambda \tilde{\beta}_g$$

If this holds, then calculating $\partial_y A$ and requiring $\partial_y A = T_{IJ}\beta^J$ forces (for some calculable constant θ)

$$3\bar{y} \cdot \Lambda - 2\lambda C_R = \gamma - \gamma^2 + \Theta \circ \beta_{\bar{y}} + \theta \tilde{\beta}_g$$

This is the Λ-equation, and allows restrictions on the form of anomalous dimensions for $\mathcal{N} = 1$ SUSY. Specialising to $\mathcal{N} = 2$ gives many more constraints at higher orders.

Supersymmetric theories

The 3-loop Anomalous Dimension

- Similar to A, one can calculate perturbatively
- $3\bar{y}\cdot\Lambda^{(1)} - 2\lambda^{(1)}C_R = \gamma^{(1)}$
- $3\bar{y}\cdot\Lambda^{(2)} - 2\lambda^{(2)}C_R = \gamma^{(2)} - \gamma^{(1)2} + \Theta^{(1)}\beta_{\bar{y}}^{(1)} + \ldots$
- $3\bar{y}\cdot\Lambda^{(3)} - 2\lambda^{(3)}C_R = \gamma^{(3)} - \gamma^{(2)}\gamma^{(1)} - \gamma^{(1)}\gamma^{(2)} + \Theta^{(2)}\beta_{\bar{y}}^{(2)} + \ldots$

Λ now plays the role of A, in that by determining Λ with arbitrary coefficients one can derive consistency conditions on $\gamma^{(3)}$. There are eight such conditions, which are satisfied by both DRED and NSVZ versions of $\gamma^{(3)}$. Reducing further to $\mathcal{N} = 2$ gives yet more conditions that are also satisfied.

Summary

- Complete calculation (up to purely gauge terms) of $A^{(4)}$ for general renormalizable QFT.
- Derived scheme-independent consistency conditions.
- Verified by comparison with both general 3-loop single gauge β-function and Standard Model 3-loop gauge β-function.
- Demonstrated restriction on the form of $\gamma^{(3)}$ via the Λ-equation; almost solvable, constraints satisfied by actual $\gamma^{(3)}$ calculation.

Summary

Further work:

- General $A^{(5)}$ calculation, starting with just scalar/fermion couplings, to predict form/consistency conditions for higher-order general β-functions. Metric no longer symmetric at this order.
- Consider other dimensions: analogous work for 6D ϕ^3 theory up to $A^{(5)}$ almost complete. 6D interesting since leading order metric is *negative*-definite, giving opposite conclusion to 4D case.

Summary

Key sources:

- I. Jack, H. Osborn, *Analogs For The c-Theorem For Four-dimensional Renormalizable Field Theories*, Nucl. Phys. B343 (1990) 647
- I. Jack, H. Osborn, *Constraints on RG flow for Four-dimensional Quantum Field Theories*, Nucl. Phys. B883 (2014) 425
- I. Jack, C. Poole, *The a-function for gauge theories*, arXiv:1411.1301v1